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VARIOUS POLYNOMIALS ASSOCIATED TO GRAPHS INVOLVING SPLICE AND LINK STRUCTURES

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ABSTRACT. Cones are the graph structures involving an universal vertex, which is adjacent to all other vertices. Splice and link structures of graphs involving cones, which are formed by using regular graphs, can be viewed in terms of joined union of graphs. The current work is about polynomials associated to adjacency, Laplacian and signless Laplacian matrix for graphs involving cones. Significantly it is noted that, results due to adjacency polynomial of splice and link structure of cones are generalizations of those due to structures involving complete graphs, which are available in the literature.

1. Introduction

One of the methods of studying a particular type of graph structures is the study related to graph operations. T. Došlić in 2005 [6], defined two new graph operations called splice and link of two graphs, which involve vertex identification among two graphs and linking two graphs with an edge, respectively. These structures are prominent, as they are observed in many chemical compounds. For example, splice of cycles serve as models of spirane molecules and models of complex molecules are built from simpler binding blocks by iterating and/or combining splice and link operations. The spectral approach to splice and link of two graphs interms of characteristic polynomials was due to F. Celik et al. in [4]. Recently, Ramane et al. [8] extended the concept of splice and link (include symmetric structures) of two graphs to any number of graphs, which is due to the fact that spiro compounds can have more than two molecular rings. Hence, study spectra and energy of these structures for certain class of graphs.

The present work is intended to study splice and link structures of graphs involving cones (can be viewed in terms of joined union), which includes the study of adjacency polynomial, Laplacian polynomial, signless Laplacian polynomial and constuction of certain class of equienergetic graphs. Notable significance is

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that, results due to adjacency polynomial of splice and link structure of cones are generalizations of those due to structures involving complete graphs in [8].

2. Preliminaries

Throughout, we consider simple, connected, undirected and labeled graphs. In a graph G on n vertices with the vertex set $\{v_1, v_2, \ldots, v_n\}$, two vertices v_i and v_j are said to be adjacent whenever there is an edge between them, the degree of a vertex v_i is the number of edges incident to it, if all the vertices have same degree say a constant r then G is an r-regulr graph.

For a graph G with n vertices, the adjacency matrix is defined as $A(G) = [a_{ij}]_{n\times n}$ in which $a_{ij} = 1$ if vertices v_i and v_j are adjacent and zero otherwise. Polynomial associated to $A(G)$ given by $\phi_A(G; x) = \big|I_n x - A(G)\big|$ is called as adjacency polynomial, roots of the equation $\phi_A(G; x) = 0$ are adjacency eigenvalues, denote them by $\lambda_1, \lambda_2, \ldots, \lambda_n$, collection of these eigenvalues constitute adjacency spectrum, aggregate of the absolute values of eigenvalues forming spectrum of a graph will be graph energy. Two graphs are said to be equienergetic, if they have same energy.

$D(G) = (d_1, d_2, \ldots, d_n)$ denote diagonal matrix of the vertex degrees in G. Laplacian matrix and signless Laplacian matrix for a graph on n vertices are $n \times n$ matrices, $L(G) = D(G) - A(G)$ and $L^+(G) = A(G) + D(G)$, respectively. Polynomial associated to $L(G)_{n\times n}$ given by $\phi_L(G; x) = \big|I_n x - L(G)\big|$ is called as Laplacian polynomial, roots of the equation $\phi_L(G; x) = 0$ are Laplacian eigenvalues, denote them by $\mu_1, \mu_2, \mu_3 \ldots, \mu_n$, collection of these eigenvalues constitute Laplacian spectrum. Analogously, polynomial associated to $L^+(G)_{n\times n}$ given by $\phi_{L^+}(G; x) = \big|I_n x - L^+(G)\big|$ is called as signless Laplacian polynomial, roots of the equation $\phi_{L^+}(G; x) = 0$ are signless Laplacian eigenvalues, denote them by $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$, collection of these eigenvalues constitute signless Laplacian spectrum. For an r-regular graph on n vertices, adjacency, Laplacian and signless Laplacian spectrum are given by,

$$\Big\{r,\ \lambda_2(G),\ \lambda_3(G), \ldots,\ \lambda_n(G)\Big\}$$

$$\Big\{0,\ \mu_2(G),\ \mu_3(G), \ldots,\ \mu_n(G)\Big\} \text{ and}$$

$$\Big\{2r,\ \gamma_2(G),\ \gamma_3(G), \ldots,\ \gamma_n(G)\Big\}, \text{ respectively.}$$

K_s denote complete graph on s vertices. For other graph theoretical definitions and terminologies, we follow the book [5]. More work on the concepts of joned union and equitable partition be seen in [1, 2, 3, 7, 10, 11]

Definition 2.1. [9] For a graph G on n vertices with the vertex set labeling $V(G) = \{v_1, v_2, \ldots, v_n\}$, the joined union of n arbitrary graphs $G_1, G_2, \ldots, G_n$ denoted by $G[G_1, G_2, \ldots, G_n]$, is the graph obtained from the union of graphs $G_1, G_2, \ldots, G_n$ by joining every vertex of G_i to each vertex of G_j whenever v_i and v_j are adjacent in G.

If $G = K_2$, we call the joined union of two graphs G_1 and G_2 as join of graphs and denote it by $G_1 \nabla G_2$. The graph $G_1 \nabla v$ can be called as a cone, in which v is the universal vertex as is adjacent to all vertices of G_1.

Definition 2.2. [8] Let $G_1, G_2, \ldots, G_p$ be p disjoint graphs and let us label p vertices, one in each $V(G_i)$ for $i = 1, 2, \ldots, p$, by v. The vertex joining graph at v or the splice of these graphs, denoted as $\vee_v[G_1, G_2, \ldots, G_p]$, is obtained by identifying the vertices v of the p graphs.

If all the graphs are same then, let us denote splice of them as:

$$\vee_v[\underbrace{G, G, \ldots, G}_{p\text{-copies}}] \text{ or } \vee_v \Big\{ \bigcup_{p\text{-copies}} G \Big\}.$$

Definition 2.3. [8] Let $G_1, G_2, \ldots, G_{2p}$ be $2p$ graphs and let us label p vertices, one in each $V(G_i)$ for $i = 1, 2, \ldots, p$, by v and other p vertices, one in each $V(G_i)$ for $i = p+1, p+2, \ldots, 2p$, by v'. The edge joining graph at vv' or the link of these graphs be denoted as $\vee^e_{vv'}[G_1, G_2, \ldots, G_{2p}]$ which is obtained by adding a new edge between the identified vertices v and v' of $2p$ graphs.

If all the graphs are same then, let us denote link of them as:

$$\vee^e_{vv'}[\underbrace{G, G, \ldots, G}_{2p\text{-copies}}] \text{ or } \Big\{ \bigcup_{p\text{-copies}} G \Big\} \vee^e_{vv'} \Big\{ \bigcup_{p\text{-copies}} G \Big\}.$$

Definition 2.4. [9] A partition $\pi : V_1, V_2, \ldots, V_m$ of the vertex set V(G) of a graph G is equitable if the number of neighbors in V_j for a vertex u in V_i is a constant q_{ij}, independent of u for all i, j $(1 \le i, j \le m)$.

It is noted that, a partition of the vertex set $V(G)$ of a graph G is the collection of subsets V_i for $i = 1, 2, \ldots, m$ of $V(G)$ such that they are pairwise disjoint and their union will be $V(G)$.
The matrix associated with an equitable partition π is called a quotient matrix, which is given by $Q = [q_{ij}]_{m \times m}$.
If we study equitable partition with respect to adjacency, Laplacian and signless Laplacian matrix, the corresponding quotient matrix is denoted by Q, Q_L and Q_{L^+}, respectively. Where

$$Q_L = \begin{cases} \sum_{j=1}^m q_{ij} & , \quad \text{if } i = j \\ -q_{ij} & , \quad \text{otherwise} \end{cases}$$

and

$$Q_{L^+} = \begin{cases} \sum_{j=1}^m q_{ij} & , \quad \text{if } i = j \\ q_{ij} & , \quad \text{otherwise.} \end{cases}$$

Theorem 2.5. [9] *If $\pi : V_1, V_2, \ldots, V_m$ is an equitable partition of a graph G then $\phi(Q; x)$ divides $\phi_A(G\ ;\ x)$.*

Theorem 2.6. [1] *If $\pi : V_1, V_2, \ldots, V_m$ is an equitable partition of a graph G then $\phi\Big(Q_L; x\Big)$ divides $\phi_L(G\ ;\ x)$.*

Theorem 2.7. [10] *If $\pi : V_1, V_2, \dots, V_m$ is an equitable partition of a graph G then $\phi\Big(Q_{L^+}; x\Big)$ divides $\phi_{L^+}(G\ ;\ x)$.*

3. Various polynomials for graphs involving splice structures

This section focuses on the adjacency, Laplacian and signless Laplacian polynomials associated to graphs involving splice structures.

3.1. Adjacency polynomial of splice structures.

Theorem 3.1. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, adjacency polynomial of the splice of graphs $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v$ at the vertex v is*

$$\begin{aligned}&\phi_A\Big(\vee_v\,[G_1 \bigtriangledown v, G_2 \bigtriangledown v]\ ;\ x\Big)\\ &= \frac{\phi_A(G_1;x)}{x-r_1}\ \frac{\phi_A(G_2;x)}{x-r_2}\Big\{x^3-(r_1+r_2)x^2+(r_1r_2-n_1-n_2)x\\ &\qquad\qquad +(r_1n_2+r_2n_1)\Big\}.\end{aligned}$$

Proof. Graphs G_1 and G_2 are regular graphs on n_1 and n_2 vertices with adjacency spectra

$$\Big\{r_1,\ \lambda_2(G_1),\ \lambda_3(G_1), \dots,\ \lambda_{n_1}(G_1)\Big\}\ \text{ and}$$

$$\Big\{r_2,\ \lambda_2(G_2),\ \lambda_3(G_2), \dots,\ \lambda_{n_2}(G_2)\Big\},\ \text{respectively.}$$

The graph structure of $\vee_v[G_1 \bigtriangledown v, G_2 \bigtriangledown v]$ involves two graphs $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v$ which are identified at v, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making n_1+n_2+1 vertices into three partite sets: $V_1 = V(G_1)$, $V_2 = \{v\}$ and $V_3 = V(G_2)$, these three partite sets lead to the quotient matrix

$$Q = \begin{bmatrix} r_1 & 1 & 0\\ n_1 & 0 & n_2\\ 0 & 1 & r_2\end{bmatrix}.$$

The polynomial associated to Q is

$$\phi(Q;x) = x^3-(r_1+r_2)x^2+(r_1r_2-n_1-n_2)x+(r_1n_2+r_2n_1).$$

Remaining part of the spectrum of $\vee_v[G_1 \bigtriangledown v, G_2 \bigtriangledown v]$ is due to the partitions V_1 and V_3 which is:

$$\Big\{\lambda_i(G_1) : i=2,3,\dots,n_1\Big\}\bigcup\Big\{\lambda_i(G_2) : i=2,3,\dots,n_2\Big\}.$$

Hence, by Theorem 2.5, result follows. ∎

Theorem 3.2. *If G is an r-regular graph on n vertices. Then, adjacency polynomial of the splice of p graphs,* $\mathcal{G} = \vee_v[\,\underbrace{G \bigtriangledown v, G \bigtriangledown v, \ldots, G \bigtriangledown v}_{p\text{-copies}}\,]$ *is*

$$\phi_A\big(\mathcal{G};\ x\big) \quad = \quad \frac{\Big(\phi_A(G;x)\Big)^p}{x-r}\ \Big\{x^2 - rx - np\Big\}.$$

Proof. Let G be an r-regular graph on n vertices with adjacency spectrum

$$\Big\{r,\ \lambda_2(G),\ \lambda_3(G), \ldots,\ \lambda_n(G)\Big\}.$$

Structure of $\mathcal{G}$ involves p-copies of $G\nabla v$ which are identified at v, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $np+1$ vertices into two partite sets: $V_1 = \bigcup_{p\text{-copies of } G} V(G)$ with p-copies of G labeled with different vertex labels and $V_2 = \{v\}$, these two partite sets lead to the quotient matrix

$$Q = \begin{bmatrix} r & 1 \\ np & 0 \end{bmatrix}.$$

The polynomial associated to Q is

$$\phi(Q;x) = x^2 - rx - np.$$

Remaining part of the spectrum of $\mathcal{G}$ is due to the partition V_1, which is:

$$\Big\{r^{(p-1)},\ \Big(\lambda_i(G)\Big)^{(p)} : i = 2, 3, \ldots, n\Big\}$$

Hence, by Theorem 2.5, result follows. □

Remark 3.3. If we take $G = K_{s-1}$ (a complete graph on $s-1$ vertices) and $p = 2$ in Theorem 3.2, we get $\mathcal{G} = \vee_v[K_{s-1} \bigtriangledown v, K_{s-1} \bigtriangledown v]$. Thus,

$$\phi_A\big(\mathcal{G};\ x\big) \quad = \quad \big(x - (s-2)\big)\ (x+1)^{2s-4}\ \Big(x^2 - (s-2)x - 2s + 2\Big).$$

Hence, Theorem 3.2 is the generalization of Theorem 3.3 and Theorem 3.4 due to F. Celik et al. in [4].

Remark 3.4. If we take $G = K_{s-1}$ (a complete graph on $s-1$ vertices) in Theorem 3.2, we get $\mathcal{G} = \vee_v[\,\underbrace{K_{s-1} \bigtriangledown v, K_{s-1} \bigtriangledown v, \ldots, K_{s-1} \bigtriangledown v}_{p\text{-copies}}\,]$. Thus,

$$\phi_A\big(\mathcal{G};\ x\big) \quad = \quad \big(x - (s-2)\big)^{p-1}\ (x+1)^{p(s-2)}\ \Big(x^2 - (s-2)x - (s-1)p\Big).$$

Hence, Theorem 3.2 is the generalization of Theorem 2.2 due to Ramane et al. in [8].

3.2. Laplacian polynomial of splice structures.

Theorem 3.5. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, Laplacian polynomial of the splice of graphs $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v$ at the vertex v is*

$$\phi_L\Big(\vee_v \left[G_1 \bigtriangledown v, G_2 \bigtriangledown v\right] \ ; \ x\Big)$$
$$= \frac{\phi_L(G_1; x-1)}{x-1} \ \frac{\phi_L(G_2; x-1)}{x-1} \ x \ \Big\{x^2 - (n_1+n_2+2)x + (n_1+n_2+1)\Big\}.$$

Proof. Graphs G_1 and G_2 are regular graphs on n_1 and n_2 vertices with Laplacian spectra

$$\Big\{0, \ \mu_2(G_1), \ \mu_3(G_1), \ldots, \ \mu_{n_1}(G_1)\Big\} \text{ and}$$

$$\Big\{0, \ \mu_2(G_2), \ \mu_3(G_2), \ldots, \ \mu_{n_2}(G_2)\Big\}, \text{ respectively.}$$

The graph structure of $\vee_v[G_1 \bigtriangledown v, G_2 \bigtriangledown v]$ involves two graphs $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v$ which are identified at v, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $n_1 + n_2 + 1$ vertices into three partite sets: $V_1 = V(G_1)$, $V_2 = \{v\}$ and $V_3 = V(G_2)$, these three partite sets lead to the quotient matrix

$$Q_L = \begin{bmatrix} 1 & -1 & 0 \\ -n_1 & n_1+n_2 & -n_2 \\ 0 & -1 & 1 \end{bmatrix}.$$

The polynomial associated to Q_L is

$$\phi(Q_L; x) = x \ \Big(x^2 - (n_1+n_2+2)x + (n_1+n_2+1)\Big).$$

Remaining part of the Laplacian spectrum of $\vee_v[G_1 \bigtriangledown v, G_2 \bigtriangledown v]$ is due to the partitions V_1 and V_3 which is:

$$\Big\{\mu_i(G_1) + 1 : i = 2, 3, \ldots, n_1\Big\} \bigcup \Big\{\mu_i(G_2) + 1 : i = 2, 3, \ldots, n_2\Big\}.$$

Hence, by Theorem 2.6, result follows. □

Theorem 3.6. *If G is an r-regular graph on n vertices. Then, Laplacian polynomial of the splice of p graphs, $\mathcal{G} = \vee_v[\underbrace{G \bigtriangledown v, G \bigtriangledown v, \ldots, G \bigtriangledown v}_{p\text{-copies}}]$ is*

$$\phi_L\big(\mathcal{G}; \ x\big) \ = \ \frac{\Big(\phi_L(G; x-1)\Big)^p}{x-1} \ x \ \Big\{x - (np+1)\Big\}.$$

Proof. Let G be an r-regular graph on n vertices with Laplacian spectrum

$$\Big\{0, \ \mu_2(G), \ \mu_3(G), \ldots, \ \mu_n(G)\Big\}.$$

Structure of $\mathcal{G}$ involves p-copies of $G\nabla v$ which are identified at v, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $np+1$ vertices into two partite sets: $V_1 = \bigcup_{p\text{-copies of } G} V(G)$ with p-copies of G labeled with different vertex labels and $V_2 = \{v\}$, these two partite sets lead to the quotient matrix

$$Q_L = \begin{bmatrix} 1 & -1 \\ -np & np \end{bmatrix}.$$

The polynomial associated to Q_L is

$$\phi(Q_L; x) = x\left\{x - (np+1)\right\}.$$

Remaining part of the Laplacian spectrum of $\mathcal{G}$ is due to the partition V_1, which is:

$$\left\{1^{(p-1)},\ \Big(\mu_i(G)+1\Big)^{(p)} : i = 2, 3, \ldots, n\right\}$$

Hence, by Theorem 2.6, result follows. □

3.3. **Signless Laplacian polynomial of splice structures.**

Theorem 3.7. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, signless Laplacian polynomial of the splice of graphs $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v$ at the vertex v is*

$$\begin{aligned}
&\phi_{L^+}\Big(\vee_v\left[G_1 \bigtriangledown v, G_2 \bigtriangledown v\right] ;\ x\Big) \\
&= \frac{\phi_{L^+}(G_1; x-1)}{x - 2r_1 - 1}\ \frac{\phi_{L^+}(G_2; x-1)}{x - 2r_2 - 1}\Bigg\{x^3 - \Big(2(r_1+r_2) + n_1 + n_2 + 2\Big)x^2 \\
&\qquad + \Big(2(n_1+n_2)\ (r_1+r_2) + 2(r_1 + r_2 + 2r_1 r_2) + n_1 + n_2 + 1\Big)x \\
&\qquad\qquad -2\Big(2r_1 r_2(n_1+n_2) + (r_1 n_1 + r_2 n_2)\Big)\Bigg\}.
\end{aligned}$$

Proof. Graphs G_1 and G_2 are regular graphs on n_1 and n_2 vertices with signless Laplacian spectra

$$\Big\{2r_1,\ \gamma_2(G_1),\ \gamma_3(G_1), \ldots,\ \gamma_{n_1}(G_1)\Big\} \text{ and}$$

$$\Big\{2r_2,\ \gamma_2(G_2),\ \gamma_3(G_2), \ldots,\ \gamma_{n_2}(G_2)\Big\}, \text{ respectively.}$$

The graph structure of $\vee_v[G_1 \bigtriangledown v, G_2 \bigtriangledown v]$ involves two graphs $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v$ which are identified at v, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $n_1 + n_2 + 1$

vertices into three partite sets: $V_1 = V(G_1)$, $V_2 = \{v\}$ and $V_3 = V(G_2)$, these three partite sets lead to the quotient matrix

$$Q_{L^+} = \begin{bmatrix} 2r_1+1 & 1 & 0 \\ n_1 & n_1+n_2 & n_2 \\ 0 & 1 & 2r_2+1 \end{bmatrix}.$$

The polynomial associated to Q_{L^+} is

$$\phi(Q_{L^+};x) = \Bigg\{ x^3 - \big(2(r_1+r_2)+n_1+n_2+2\big)x^2 + \Big(2(n_1+n_2)\ (r_1+r_2)+2(r_1 + r_2 + 2r_1r_2) + n_1+n_2+1\Big)x - 2\Big(2r_1r_2(n_1+n_2) + (r_1n_1+r_2n_2)\Big)\Bigg\}.$$

Remaining part of the spectrum of $\vee_v[G_1 \bigtriangledown v, G_2 \bigtriangledown v]$ is due to the partitions V_1 and V_3 which is:

$$\Big\{\gamma_i(G_1)+1 : i = 2,3,\ldots,n_1\Big\} \bigcup \Big\{\gamma_i(G_2)+1 : i=2,3,\ldots,n_2\Big\}.$$

Hence, by Theorem 2.7, result follows. □

Theorem 3.8. *If G is an r-regular graph on n vertices. Then, signless Laplacian polynomial of the splice of p graphs,* $\mathcal{G} = \vee_v[\underbrace{G \bigtriangledown v, G \bigtriangledown v, \ldots, G \bigtriangledown v}_{p\text{-copies}}]$ *is*

$$\phi_{L^+}(\mathcal{G};\ x) \quad = \quad \frac{\Big(\phi_{L^+}(G; x-1)\Big)^p}{x-2r-1}\Big\{x^2 - (2r+np+1)x + 2rnp\Big\}$$

Proof. Let G be an r-regular graph on n vertices with Laplacian spectrum

$$\Big\{2r,\ \gamma_2(G),\ \gamma_3(G),\ldots,\ \gamma_n(G)\Big\}.$$

Structure of $\mathcal{G}$ involves p-copies of $G\nabla v$ which are identified at v, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $np+1$ vertices into two partite sets: $V_1 = \bigcup_{p\text{-copies of } G} V(G)$ with p-copies of G labeled with different vertex labels and $V_2 = \{v\}$, these two partite sets lead to the quotient matrix

$$Q_{L^+} = \begin{bmatrix} 2r+1 & 1 \\ np & np \end{bmatrix}.$$

The polynomial associated to Q_{L^+} is

$$\phi(Q_{L^+};x) = \Big\{x^2 - (2r+np+1)x + 2rnp\Big\}.$$

Remaining part of the signless Laplacian spectrum of $\mathcal{G}$ is due to the partition V_1, which is:

$$\Big\{\Big(2r+1\Big)^{(p-1)},\ \Big(\gamma_i(G)+1\Big)^{(p)} : i = 2,3,\ldots,n\Big\}.$$

Hence, by Theorem 2.7, result follows. □

4. Various polynomials for graphs involving link structures

This section focuses on the adjacency, Laplacian and signless Laplacian polynomials associated to graphs involving link structures.

4.1. **Adjacency polynomial of link structures.**

Theorem 4.1. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, adjacency polynomial of the link of graphs $2p$ graphs, $\mathcal{G} = \Big\{\bigcup_{p\text{-copies}} G_1 \triangledown v\Big\} \ \vee^{e}_{vv'} \ \Big\{\bigcup_{p\text{-copies}} G_2 \triangledown v'\Big\}$ involving p-copies of $G_1 \triangledown v$ spliced at v and p-copies of $G_2 \triangledown v'$ spliced at v', linked through $e = vv'$ is*

$$\begin{aligned}\phi_A\big(\mathcal{G};\ x\big) \ &= \ \frac{\Big(\phi_A(G_1;x)\Big)^p}{x-r_1}\ \frac{\Big(\phi_A(G_2;x)\Big)^p}{x-r_2}\Big\{x^4-(r_1+r_2)x^3+(r_1r_2\\ &\qquad -n_1p-n_2p-1)x^2+(n_1r_2p+n_2r_1p+r_1+r_2)x\\ &\qquad +(n_1n_2p^2-r_1r_2)\Big\}.\end{aligned}$$

Proof. Graphs G_1 and G_2 are regular graphs on n_1 and n_2 vertices with adjacency spectra

$$\Big\{r_1,\ \lambda_2(G_1),\ \lambda_3(G_1),\ldots,\ \lambda_{n_1}(G_1)\Big\} \text{ and}$$
$$\Big\{r_2,\ \lambda_2(G_2),\ \lambda_3(G_2),\ldots,\ \lambda_{n_2}(G_2)\Big\}, \text{ respectively.}$$

The graph structure of $\mathcal{G} = \Big\{\bigcup_{p\text{-copies}} G_1 \triangledown v\Big\} \ \vee^{e}_{vv'} \ \Big\{\bigcup_{p\text{-copies}} G_2 \triangledown v'\Big\}$ involves p-copies of $G_1 \triangledown v$ spliced at v and p-copies of $G_2 \triangledown v$ spliced at v', linked through $e = vv'$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $n_1p + n_2p + 2$ vertices into four partite sets: $V_1 = \bigcup_{p\text{-copies of } G_1} V(G_1)$, $V_2 = \{v\}$, $V_3 = \{v'\}$ and $V_4 = \bigcup_{p\text{-copies of } G_2} V(G_2)$, these four partite sets lead to the quotient matrix

$$Q = \begin{bmatrix} r_1 & 1 & 0 & 0\\ n_1p & 0 & 1 & 0\\ 0 & 1 & 0 & n_2p\\ 0 & 0 & 1 & r_2\end{bmatrix}.$$

The polynomial associated to Q is

$$\begin{aligned}\phi(Q;x) \ &= \ \Big\{x^4-(r_1+r_2)x^3+(r_1r_2-n_1p-n_2p-1)x^2+(n_1r_2p+n_2r_1p\\ &\qquad +r_1+r_2)x+(n_1n_2p^2-r_1r_2)\Big\}.\end{aligned}$$

Remaining part of the spectrum of $\mathcal{G}$ is due to the partitions V_1 and V_4 which is:

$$\Big\{r_1^{(p-1)},\ \Big(\lambda_i(G_1)\Big)^{(p)} : i = 2,3,\ldots,n_1\Big\} \bigcup \Big\{r_2^{(p-1)},\ \Big(\lambda_i(G_2)\Big)^{(p)} : i = 2,3,\ldots,n_2\Big\}.$$

Hence, by Theorem 2.5, result follows. □

Corollary 4.2. *If G is an r-regular graphs on n vertices. Then, adjacency polynomial of the link of $2p$ graphs, $\mathcal{G} = \Big\{ \bigcup_{p\text{-copies}} G \bigtriangledown v \Big\} \vee^{e}_{vv'} \Big\{ \bigcup_{p\text{-copies}} G \bigtriangledown v' \Big\}$, involving p-copies of $G \bigtriangledown v$ spliced at v and p-copies of $G \bigtriangledown v'$ spliced at v', linked through $e = vv'$ is*

$$\begin{aligned}\phi_A\big(\mathcal{G};\ x\big) &= \frac{\Big(\phi_A(G;x)\Big)^{2p}}{(x-r)^2} \Big\{ x^4 - 2rx^3 + (r^2 - 2np - 1)x^2 \\ &\qquad + 2r(np+1)x + (n^2p^2 - r^2) \Big\}.\end{aligned}$$

Proof. Result follows from Theorem 4.1 by taking $G_1 = G_2 = G$ and hence, $n_1 = n_2 = n$, $r_1 = r_2 = r$. □

Remark 4.3. If we take $G = K_{s-1}$ (a complete graph on $s-1$ vertices) and $p = 1$ in Corollary 4.2 we get $\mathcal{G} = [K_{s-1} \bigtriangledown v] \ \vee^{e}_{vv'} \ [K_{s-1} \bigtriangledown v']$. Thus,

$$\begin{aligned}\phi_A\big(\mathcal{G};\ x\big) &= (x+1)^{2s-4} \Big\{ x^4 - (2s-4)x^3 + (s^2 - 6s + 5)x^2 + 2s(s-2)x \\ &\qquad + (2s-3) \Big\}.\end{aligned}$$

Hence, Corollary 4.2 is the generalization of Theorem 3.11 due to F. Celik et al. in [4].

Remark 4.4. If we take $G = K_{s-1}$ (a complete graph on $s-1$ vertices) in Corollary 4.2, we get $\mathcal{G} = \Big\{ \bigcup_{p\text{-copies}} K_{s-1} \bigtriangledown v \Big\} \vee^{e}_{vv'} \Big\{ \bigcup_{p\text{-copies}} K_{s-1} \bigtriangledown v' \Big\}$. Thus,

$$\begin{aligned}\phi_A\big(\mathcal{G};\ x\big) &= \big(x - (s-2)\big)^{2p-2} (x+1)^{2p(s-2)} \Big\{ x^4 - (2s-4)x^3 \\ &\quad + (s^2 - 2sp - 4s + 2p + 3)x^2 + 2(s-2)\big[1 + p(s-1)\big]x \\ &\quad + \big[p^2(s-1)^2 - (s-2)^2\big] \Big\}.\end{aligned}$$

Hence, Corollary 4.2 is the generalization of Theorem 3.2 due to Ramane et al. in [8].

Corollary 4.5. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, adjacency polynomial of the link of graphs two graphs $\mathcal{G} = [G_1 \bigtriangledown v] \ \vee^{e}_{vv'} \ [G_2 \bigtriangledown v']$, involving $G_1 \bigtriangledown v$ and $G_2 \bigtriangledown v'$, linked through $e = vv'$ is*

$$\begin{aligned}\phi_A\big(\mathcal{G};\ x\big) &= \frac{\phi_A(G_1;x)}{x - r_1} \frac{\phi_A(G_2;x)}{x - r_2} \Big\{ x^4 - (r_1 + r_2)x^3 + (r_1r_2 - n_1 - n_2 \\ &\qquad - 1)x^2 + (n_1r_2 + n_2r_1 + r_1 + r_2)x + (n_1n_2 - r_1r_2) \Big\}.\end{aligned}$$

Proof. Result follows from Theorem 4.1 by taking $p = 1$. □

4.2. **Laplacian polynomial of link structures.**

Theorem 4.6. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, Laplacian polynomial of the link of $2p$ graphs $\mathcal{G} = \Big\{ \bigcup_{p\text{-copies}} G_1 \bigtriangledown v \Big\} \vee^{e}_{vv'} \Big\{ \bigcup_{p\text{-copies}} G_2 \bigtriangledown v' \Big\}$, involving p-copies of $G_1 \bigtriangledown v$ spliced at v and p-copies of $G_2 \bigtriangledown v'$ spliced at v', linked through $e = vv'$ is*

$$\begin{aligned}\phi_L\big(\mathcal{G};\ x\big) &= \frac{\Big(\phi_L(G_1; x-1)\Big)^p}{x-1} \frac{\Big(\phi_L(G_2; x-1)\Big)^p}{x-1}\, x \Big\{ x^3 - (n_1p + n_2p + 4)x^2 \\ &\quad + (n_1n_2p^2 + 2n_1p + 2n_2p + 5)x - (n_1p + n_2p + 2) \Big\}.\end{aligned}$$

Proof. Graphs G_1 and G_2 are regular graphs on n_1 and n_2 vertices with Laplacian spectrum

$$\Big\{ 0,\ \mu_2(G_1),\ \mu_3(G_1), \ldots,\ \mu_{n_1}(G_1) \Big\} \text{ and}$$

$$\Big\{ 0,\ \mu_2(G_2),\ \mu_3(G_2), \ldots,\ \mu_{n_2}(G_2) \Big\}, \text{ respectively.}$$

The graph structure of $\mathcal{G} = \Big\{ \bigcup_{p\text{-copies}} G_1 \bigtriangledown v \Big\} \vee^{e}_{vv'} \Big\{ \bigcup_{p\text{-copies}} G_2 \bigtriangledown v' \Big\}$ involves p-copies of $G_1 \bigtriangledown v$ spliced at v and p-copies of $G_2 \bigtriangledown v$ spliced at v', linked through $e = vv'$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $n_1p + n_2p + 2$ vertices into four partite sets: $V_1 = \bigcup_{p\text{-copies of } G_1} V(G_1)$, $V_2 = \{v\}$, $V_3 = \{v'\}$ and $V_4 = \bigcup_{p\text{-copies of } G_2} V(G_2)$, these four partite sets lead to the quotient matrix

$$Q_L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -n_1p & n_1p+1 & -1 & 0 \\ 0 & -1 & n_2p+1 & -n_2p \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The polynomial associated to Q_L is

$$\begin{aligned}\phi(Q_L; x) &= x \Big\{ x^3 - (n_1p + n_2p + 4)x^2 + (n_1n_2p^2 + 2n_1p + 2n_2p + 5)x \\ &\quad - (n_1p + n_2p + 2) \Big\}.\end{aligned}$$

Remaining part of the Laplacian spectrum of $\mathcal{G}$ is due to the partitions V_1 and V_4 which is:

$$\begin{aligned}&\Big\{ 1^{2(p-1)} \Big\} \bigcup \Big\{ \Big(\mu_i(G_1) + 1\Big)^{(p)} : i = 2, 3, \ldots, n_1 \Big\} \\ &\quad \bigcup \Big\{ \Big(\mu_i(G_2) + 1\Big)^{(p)} : i = 2, 3, \ldots, n_2 \Big\}.\end{aligned}$$

Hence, by Theorem 2.6, result follows. $\square$

4.3. Signless Laplacian polynomial of link structures.

Theorem 4.7. *If G_1 and G_2 are r_1 and r_2 regular graphs on n_1 and n_2 vertices, respectively. Then, signless Laplacian polynomial of the link of $2p$ graphs $\mathcal{G} = \left\{\bigcup_{p\text{-copies}} G_1 \bigtriangledown v\right\} \vee^{e}_{vv'} \left\{\bigcup_{p\text{-copies}} G_2 \bigtriangledown v'\right\}$, involving p-copies of $G_1 \bigtriangledown v$ spliced at v and p-copies of $G_2 \bigtriangledown v'$ spliced at v', linked through $e = vv'$ is*

$$\begin{aligned}\phi_{L^+}\big(\mathcal{G};\ x\big) &= \frac{\Big(\phi_{L^+}(G_1; x-1)\Big)^p}{x-2r_1-1}\ \frac{\Big(\phi_{L^+}(G_2; x-1)\Big)^p}{x-2r_2-1} \\ &\qquad\qquad \Big\{x^4 - C_1x^3 + C_2x^2 - C_3x + C_4\Big\},\end{aligned}$$

where

$$\begin{aligned} C_1 &= 2(r_1+r_2) + p(n_1+n_2) + 4 \\ C_2 &= n_1n_2p^2 + 2p(n_1r_1 + n_2r_2 + n_1r_2 + n_2r_1) + 4r_1r_2 + 2p(n_1+n_2) \\ &\quad +6(r_1+r_2) + 4 \\ C_3 &= 2n_1n_2p(r_1p + r_2) + 4r_1r_2p(n_1+n_2) + 4p(n_1r_1 + n_2r_2) \\ &\quad +2p(n_1r_2 + n_2r_1) + 8r_1r_2 + p(n_1+n_2) + 4(r_1+r_2) + 2 \\ \textit{and}\ \ C_4 &= 2p\Big(2n_1n_2r_1r_2p + 2r_1r_2(n_1+n_2) + n_1r_1 + n_2r_2\Big). \end{aligned}$$

Proof. Graphs G_1 and G_2 are regular graphs on n_1 and n_2 vertices with signless Laplacian spectra

$$\Big\{2r_1,\ \gamma_2(G_1),\ \gamma_3(G_1), \ldots,\ \gamma_{n_1}(G_1)\Big\} \ \text{ and}$$

$$\Big\{2r_2,\ \gamma_2(G_2),\ \gamma_3(G_2), \ldots,\ \gamma_{n_2}(G_2)\Big\}, \ \text{respectively.}$$

The graph structure of $\mathcal{G} = \left\{\bigcup_{p\text{-copies}} G_1 \bigtriangledown v\right\} \vee^{e}_{vv'} \left\{\bigcup_{p\text{-copies}} G_2 \bigtriangledown v'\right\}$ involves p-copies of $G_1 \bigtriangledown v$ spliced at v and p-copies of $G_2 \bigtriangledown v$ spliced at v', linked through $e = vv'$, is embedded with the structure of joined union, which can be viewed with the proper partition of the vertex set. Making $n_1p + n_2p + 2$ vertices into four partite sets: $V_1 = \bigcup_{p\text{-copies of } G_1} V(G_1)$, $V_2 = \{v\}$, $V_3 = \{v'\}$ and $V_4 = \bigcup_{p\text{-copies of } G_2} V(G_2)$, these four partite sets lead to the quotient matrix

$$Q_{L^+} = \begin{bmatrix} 2r_1+1 & 1 & 0 & 0 \\ n_1p & n_1p+1 & 1 & 0 \\ 0 & 1 & n_2p+1 & n_2p \\ 0 & 0 & 1 & 2r_2+1 \end{bmatrix}.$$

The polynomial associated to Q_{L^+} is

$$\phi(Q_{L^+}; x) \ = \ \Big\{x^4 - C_1x^3 + C_2x^2 - C_3x + C_4\Big\},$$

where

$$\begin{aligned}
C_1 &= 2(r_1+r_2)+p(n_1+n_2)+4\\
C_2 &= n_1n_2p^2+2p(n_1r_1+n_2r_2+n_1r_2+n_2r_1)+4r_1r_2+2p(n_1+n_2)\\
&\quad +6(r_1+r_2)+4\\
C_3 &= 2n_1n_2p(r_1p+r_2)+4r_1r_2p(n_1+n_2)+4p(n_1r_1+n_2r_2)\\
&\quad +2p(n_1r_2+n_2r_1)+8r_1r_2+p(n_1+n_2)+4(r_1+r_2)+2\\
\text{and } C_4 &= 2p\Big(2n_1n_2r_1r_2p+2r_1r_2(n_1+n_2)+n_1r_1+n_2r_2\Big).
\end{aligned}$$

Remaining part of the signless Laplacian spectrum of $\mathcal{G}$ is due to the partitions V_1 and V_4 which is:

$$\left\{\Big(2r_1+1\Big)^{(p-1)},\ \Big(\gamma_i(G_1)+1\Big)^{(p)}: i=2,3,\ldots,n_1\right\}\bigcup$$
$$\left\{\Big(2r_2+1\Big)^{(p-1)},\ \Big(\gamma_i(G_2)+1\Big)^{(p)}: i=2,3,\ldots,n_2\right\}.$$

Hence, by Theorem 2.7, result follows. □

5. Construction of equienergetic graphs

Corollary 5.1. *If G_a, G_b are r-regular equienergetic graphs with n vertices and G_1 is an r_1-regular graph on n_1 vertices. Then, $\vee_v[G_1 \triangledown v, G_a \triangledown v]$ and $\vee_v[G_1 \triangledown v, G_b \triangledown v]$ are equienergetic.*

Corollary 5.2. *If G_a, G_b are r-regular equienergetic graphs with n vertices. Then, $\vee_v[\underbrace{G_a \triangledown v, G_a \triangledown v, \ldots, G_a \triangledown v}_{p\text{-copies}}]$ and $\vee_v[\underbrace{G_b \triangledown v, G_b \triangledown v, \ldots, G_b \triangledown v}_{p\text{-copies}}]$ are equienergetic.*

Corollary 5.3. *If G_a, G_b are r-regular equienergetic graphs with n vertices and G_1 is an r_1-regular graph on n_1 vertices. Then,*

- $\left\{\bigcup_{p\text{-copies}} G_1 \triangledown v\right\} \vee^e_{vv'} \left\{\bigcup_{p\text{-copies}} G_a \triangledown v'\right\}$, *involving p-copies of $G_1 \triangledown v$ spliced at v and p-copies of $G_a \triangledown v'$ spliced at v', linked through $e = vv'$*

and

- $\left\{\bigcup_{p\text{-copies}} G_1 \triangledown v\right\} \vee^e_{vv'} \left\{\bigcup_{p\text{-copies}} G_b \triangledown v'\right\}$, *involving p-copies of $G_1 \triangledown v$ spliced at v and p-copies of $G_b \triangledown v'$ spliced at v', linked through $e = vv'$*

are equienergetic.

Corollary 5.4. *If G_a, G_b are r-regular equienergetic graphs with n vertices. Then,*

- $\left\{\bigcup_{p\text{-copies}} G_a \triangledown v\right\} \vee^e_{vv'} \left\{\bigcup_{p\text{-copies}} G_a \triangledown v'\right\}$, *involving p-copies of $G_a \triangledown v$ spliced at v and other p-copies at v', linked through $e = vv'$*

and

- $\left\{\bigcup_{p\text{-copies}} G_b \triangledown v\right\} \vee^e_{vv'} \left\{\bigcup_{p\text{-copies}} G_b \triangledown v'\right\}$, *involving p-copies of $G_b \triangledown v$ spliced at v and other p-copies spliced at v', linked through $e = vv'$*

are equienergetic.

6. Conclusions

The study of polynomials associated to various graph matrices for splice and link structures involving cone structures become easy with the concept of equitable partition, as the splice and link structures can be viewed in terms of joined union of graphs.

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