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A STUDY OF GALOIS COVERING THROUGH THE SHEETS OF THE COVERING

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Abstract. Let $Deck_X(\tilde{X})$ denote the set of all covering transformations of a covering $p : (\tilde{X}, \tilde{x}) \to (X, x)$, where (X, x) is a path connected, locally path connected pointed topological spaces, then $Deck_X(\tilde{X})$ is a group [6].
In this paper we study
i) the Galois covering through the group $Deck_X(\tilde{X})$; also
ii) the Galois covering through the sheets of the covering; finally
iii) some applications of Galois covering.

1. Introduction and preliminaries

Throughout this paper we assume that (X, x) is a pointed topological space and maps are base point preserving continuous surjective maps. For simplicity we write X in place of (X, x).
Let $\tilde{X}$ be a connected space , X be a space and $p : \tilde{X} \to X$ be a continuous map. If for every $x \in$ X has a path connected open neighborhood U such that $p^{-1}(U)$ is open in $\tilde{X}$ and each component of $p^{-1}(U)$ is mapped homeomorphically onto U by p , then p is called a covering map and the pair $(\tilde{X}, p)$ is called a covering space of X.
Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x} \in \tilde{X}$ and $p(\tilde{x}) = x$. Then, for any path α in X with initial point x, there exists a unique path β in $\tilde{X}$ with initial point $\tilde{x}$ such that $p\beta = \alpha$.
Let $(\tilde{X}, p)$ be a covering space of X and $x \in X$. For any point $\tilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi_1(X, x)$ we define $\tilde{x}\alpha \in p^{-1}(x)$ as follows:
From the above there exists a unique path class $\tilde{\alpha}$ in $\tilde{X}$ such that $p_*(\tilde{\alpha}) = \alpha$ and the initial point of $\tilde{\alpha}$ is the point $\tilde{x}$. Define $\tilde{x}\alpha$ to be the terminal point of the path class $\tilde{\alpha}$. Then since $(\tilde{x}\alpha)\beta = \tilde{x}(\alpha\beta)$ and $\tilde{x}e = \tilde{x}$,
$\Rightarrow \pi_1(X, x)$ is a group acting from right on the set $p^{-1}(x)$, also $\pi_1(X, x)$ acts transitively on the set $p^{-1}(x)$. Let $\tilde{x_0}, \tilde{x_1} \in p^{-1}(x)$. Since $\tilde{X}$ is path connected , there exists a path class $\tilde{\alpha}$ in $\tilde{X}$ with initial point $\tilde{x_0}$ and terminal point $\tilde{x_1}$.

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Let $p_*(\tilde{\alpha}) = \alpha$. Then α is an equivalence class of closed paths and obviously $\tilde{x_0}\,\alpha = \tilde{x_1}$.
$\Rightarrow \pi_1(X, x)$ acts transitively on the set $p^{-1}(x)$.
Thus the set $p^{-1}(x)$ is a homogeneous right $\pi_1(X, x)$-space. From the definition we see that for any point $\tilde{x} \in p^{-1}(x)$, the isotropy subgroup corresponding to this point is precisely the subgroup $p_*\pi_1(\tilde{X}, \tilde{x})$ of $\pi_1(X, x)$.
Hence $p^{-1}(x)$ is isomorphic to the space of cosets $\pi_1(X, x)/p_*\pi_1(\tilde{X}, \tilde{x})$, and the number of sheets of the covering is equal to the index of the subgroup $p_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$.
If X is simply connected , then the fundamental group $\pi_1(X, x)$ is trivial and index of $p_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$ is 1. So $(\tilde{X}, p)$ is an one-sheeted covering space of X and therefore p is a homeomorphism.
Similarly if $\tilde{X}$ is simply connected, then $\pi_1(\tilde{X}, \tilde{x})$ is trivial and the index of $p_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$ is equal to the order of $\pi_1(X, x)$.
A covering transformation of a covering space $(\tilde{X}, p)$ of X is a homeomorphism $h : \tilde{X} \to \tilde{X}$ such that $ph = p$. The set of all covering transformations of $(\tilde{X}, p)$ form a group denoted by $Deck_X(\tilde{X})$.
The motivation of the work is derived from the following : properties of $\pi_1(X, x_0)$ [Massey(1984),Munkers (1975),Spanier(1978) and Adhikari(2016)].

Definition 1.1. Let $\tilde{X}$ be a path connected space and let $(\tilde{X}, p)$ be a covering space of a locally path connected space X. Then $(\tilde{X}, p)$ is a regular covering of X iff $p_*\pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$.

Definition 1.2. Let $(\tilde{X}, p)$ be a covering space of a locally path connected space X. If the group $Deck_X(\tilde{X})$ acts transitively on the set $p^{-1}(x)$, for every $x \in X$, then $(\tilde{X}, p)$ is called Galois.

Lemma 1.3. *Let G be a group and H be a subgroup of G of index 2. Then H is a normal subgroup of G.*

Proof. Suppose H is a subgroup of G of index 2. Then there are only two cosets of G relative to H. Let $a \in G - H$. Then G can be decomposed into the cosets H, aH or H,Ha, implying H commutes with a. Since $Hh = hH$ for any $h \in H$, H commutes with every element of G. Hence H is normal in G. □

Lemma 1.4. *Let G be a finite group . Let H be a subgroup of G of index p, where p is the smallest prime dividing the order of G. Then H is a normal subgroup of G.*

Proof. Suppose H is a subgroup of G with $|G|$ finite and $|G : H| = p$ where p is the smallest prime dividing $|G|$. Let G acts on the set L of left cosets of H in G by left and $\phi : G \to S_p$ be the homomorphism induced by this action. Now if $g \in ker\phi$, then $gxH = xH$ for each $x \in G$, and in particular $gH = H$ whence $g \in H$. Thus $K = ker\phi$ is a normal subgroup of H(being contained in H and normal in G). By the first isomorphism theorem, G/K is isomorphic to a subgroup of S_p, and consequently $|G/K| = |G : K|$ must divide $p!$; moreover, any divisor of $|G:K|$ must also divide $|G| = |G:K||K|$, and because p is the smallest divisor of G other than 1, the only possibilities are $|G:K| = p$ or $|G:K| = 1$. But $|G:K| = |G:H||H:K| = p|H:K| \geqslant p$, which implies $|G:K| = p$, and consequently $|H:K| = 1$, so that $H = K$, from which it follows that H is normal in G. □

Lemma 1.5. *Let G be a group of order pn, where p is a prime, n is a positive integer and $p > n$. If H is a subgroup of G of order p, then H is a normal subgroup of G.*

Proof. Let L be the set of all left cosets of H in G.Then $|L| = |G:H| = |G|/|H| = pn/p = n$. L is a (left)$G$-set by the action $g(aH) = ga(H)$ for all $g \in G$ and $aH \in L$. Let $\phi : G \to S_n$ be the homomorphism induced by this action. Now $ker\phi$ is a subset of H. Since $|H| = p$, either $ker\phi = \{e\}$ or $ker\phi = H$. If $ker\phi = \{e\}$ then G is isomorphic to a subgroup of S_n, this implies that $|G|$ divides $|S_n|$ that is $pn|n!$. Therefore $p|(n-1)!$ which is not true as $p > n$. Hence $ker\phi = H$, therefore H is a normal subgroup of G. □

Lemma 1.6. *Let G be a group of order p^n,where p is a prime and n is a positive integer. Then any subgroup of order p^{n-1} is normal in G.*

Proof. Let $n = 1$. Then G is a cyclic group of prime order p and hence every subgroup of G is normal in G. So the result is true for $n = 1$. Suppose it is true for all groups of order p^k for $1 \leqslant k < n$. Let H be a subgroup of order p^{n-1}.Consider $N(H)$(the normalizer of H). If $H \neq N(H)$, then $|N(H)| > p^{n-1}$. Thus $N(H) = p^n$,so $N(H) = G$. Hence H is normal in G.Suppose $H = N(H)$, then the center of G,$Z(G)$, is a subset of H and $Z(G) \neq \{e\}$. Hence there exists $a \in Z(G)$ such that $o(a) = p$. Let $K =< a >$. Then K is a normal subgroup of G of order p. Now $|H/K| = p^{n-2}$ and $|G/K| = p^{n-1}$. Thus by induction hypothesis H/K is a normal subgroup of G/K. Hence H is a normal subgroup of G.

□

2. STUDY OF THE GALOIS COVERING THROUGH THE GROUP $Deck_X(\tilde{X})$.

Proposition 2.1. *Let $(\tilde{X}, p)$ be a path connected covering space of a locally path connected space X and $p(\tilde{x_1}) = p(\tilde{x_2}) = x$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ and $x \in X$. Then $p_*\pi_1(\tilde{X}, \tilde{x_1})$ and $p_*\pi_1(\tilde{X}, \tilde{x_2})$ are conjugate subgroups of $\pi_1(X, x)$ iff there exists a $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$.*

Proof. Let there exists a $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$. Then we have $p\phi = p$. Now $p(\tilde{x_1}) = p(\tilde{x_2}) = x$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ and $x \in X$, then $p_*\pi_1(\tilde{X}, \tilde{x_1}) = p_*\phi_*\pi(\tilde{X}, \tilde{x_1}) \subset p_*\pi_1(\tilde{X}, \tilde{x_2})$ as $\phi(\tilde{x_1}) = \tilde{x_2}$. Because $\phi(\tilde{x_1})$ and $\tilde{x_2}$ lie in the same fiber of $p : \tilde{X} \to X$, it follows that $p_*\pi_1(\tilde{X}, \tilde{x_1})$ and $p_*\pi_1(\tilde{X}, \tilde{x_2})$ are conjugate subgroup of $\pi_1(X, x)$. Conversely suppose that $p_*\pi(\tilde{X}, \tilde{x_1})$ and $p_*\pi(\tilde{X}, \tilde{x_2})$ are conjugate subgroup of $\pi(X, x)$ such that $p(\tilde{x_1}) = p(\tilde{x_2}) = x$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ and $x \in X$. As $(\tilde{X}, p)$ is a path connected covering space of a locally path connected space X, then by the lifting theorem there exists a $\phi \in Deck_X(\tilde{X})$ such that $p\phi = p$ that is $\phi(\tilde{x_1}) = \tilde{x_2}$.

□

Proposition 2.2. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be two covering spaces of a locally path connected space X .Then these two coverings are isomorphic iff there exists $h \in Deck_X(\tilde{X})$, where $h : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 h = p_1$.*

Proof. This follows from definition of homeomorphims of covering spaces and path lifting property.

□

Theorem 2.3. *Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be two covering spaces of a path connected space X. Then there exists an isomorphism ϕ from $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ if and only if $p_{1*}\pi_1(\tilde{X}_1, \tilde{x_1}) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x_2})$.*

Proof. Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be two covering spaces of a path connected space X. A homeomorphism ϕ from $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ is an isomorphism if there exists a homeomorphism ψ from $(\tilde{X}_2, p_2)$ into $(\tilde{X}_1, p_1)$ such that both compositions $\psi\phi$ and $\phi\psi$ are identity maps. Hence there exists an isomorphism ϕ from $(\tilde{X}_1, p_1)$ into $(\tilde{X}_2, p_2)$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ if and only if $p_{1*}\pi_1(\tilde{X}_1, \tilde{x_1}) \subset p_{2*}\pi_1(\tilde{X}_2, \tilde{x_2})$ and $p_{2*}\pi_1(\tilde{X}_2, \tilde{x_2}) \subset p_{1*}\pi_1(\tilde{X}_1, \tilde{x_1})$ that is if and only if $p_{1*}\pi_1(\tilde{X}_1, \tilde{x_1}) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x_2})$. □

Corollary 2.4. *If $(\tilde{X}, p)$ be a covering space of a path connected space X such that $p(\tilde{x_1}) = p(\tilde{x_2}) = x$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ and $x \in X$, then there exists $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ if and only if $p_*\pi_1(\tilde{X}, \tilde{x_1}) = p_*\pi_1(\tilde{X}, \tilde{x_2})$.*

Proof. This follows from **Theorem 2.3**.

□

Lemma 2.5. *Let $(\tilde{X}, p)$ is an universal covering space of a locally path connected space X, then the group $Deck_X(\tilde{X})$ is isomorphic to the fundamental group $\pi_1(X, x)$, and the number of sheets of the covering is equal to the order of the fundamental group $\pi_1(X, x)$.*

Proof. Let $Deck_X(\tilde{X}) = G$. Then G acts freely and properly discontinuously on $\tilde{X}$. Let $q : \tilde{X} \to \tilde{X}/G$ be the quotient map. Then there exists a continuous map $h : \tilde{X}/G \to X$ for which $hq = p$. Now since h is a bijection and hence h is a homeomorphism. Hence $\pi_1(\tilde{X}/G, [\tilde{x}])$ is isomorphic to $\pi_1(X, x)$. Hence there exists a surjective homomorphism $k : \pi_1(X, x) \to G$ whose kernel is $p_*\pi_1(\tilde{X}, \tilde{x})$. Hence by first isomorphism theorem $G = Deck_X(\tilde{X})$ is isomorphic to $\pi_1(X, x)/p_*\pi(\tilde{X}, \tilde{x})$. As $\tilde{X}$ is simply connected, so $p_*\pi_1(\tilde{X}, \tilde{x})$ is trivial, it follows that the group $Deck_X(\tilde{X})$ is isomorphic to $\pi_1(X, x)$. As order of $Deck_X(\tilde{X})$ is the number of sheets of the covering, hence the number of sheets of the covering is equal to the order of the fundamental group $\pi_1(X, x)$.

□

Proposition 2.6. *Let $p : \tilde{X} \to X$ be a covering map such that $p(\tilde{x_1}) = p(\tilde{x_2})$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$. Then p is regular iff $p_*\pi_1(\tilde{X}, \tilde{x_1}) = p_*\pi_1(\tilde{X}, \tilde{x_2})$.*

Proof. Let $p : \tilde{X} \to X$ be a covering map such that $p(\tilde{x_1}) = p(\tilde{x_2})$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$. Then p is regular iff $Deck_X(\tilde{X})$ acts transitively on $p^{-1}(x)$ iff there exists an isomorphism ϕ of $(\tilde{X}, \tilde{x_1})$ onto $(\tilde{X}, \tilde{x_2})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ iff $p_*\pi_1(\tilde{X}, \tilde{x_1}) = p_*\pi_1(\tilde{X}, \tilde{x_2})$.

□

Proposition 2.7. *Let $(\tilde{X_1}, \tilde{x_1})$ and $(\tilde{X_2}, \tilde{x_2})$ be two covering spaces of a locally path connected space X with covering maps p_1 and p_2 respectively. Then there exists a homeomorphism ϕ from $(\tilde{X_1}, \tilde{x_1})$ into $(\tilde{X_2}, \tilde{x_2})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ if and only if $p_{1*}\pi_1(\tilde{X_1}, \tilde{x_1}) = p_{2*}\pi_1(\tilde{X_2}, \tilde{x_2})$.*

Proof. Let $(\tilde{X_1}, \tilde{x_1})$ and $(\tilde{X_2}, \tilde{x_2})$ be two covering spaces of a locally path connected space X. Then there exists a homeomorphism ϕ from $(\tilde{X_1}, \tilde{x_1})$ into $(\tilde{X_2}, \tilde{x_2})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ if and only if $p_{1*}\pi_1(\tilde{X_1}, \tilde{x_1}) \subset p_{2*}\pi_1(\tilde{X_2}, \tilde{x_2})$ and $p_{2*}\pi_1(\tilde{X_2}, \tilde{x_2}) \subset p_{1*}\pi_1(\tilde{X_1}, \tilde{x_1})$ iff $p_{1*}\pi_1(\tilde{X_1}, \tilde{x_1}) = p_{2*}\pi_1(\tilde{X_2}, \tilde{x_2})$.

□

Theorem 2.8. *Let $(\tilde{X}, p)$ be an universal covering space of a locally path connected space X and $p(\tilde{x_1}) = p(\tilde{x_2}) = x$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ and $x \in X$. Then there exists $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ i.e $Deck_X(\tilde{X})$ acts transitively on the set $p^{-1}(x)$.*

Proof. Since $(\tilde{X}, p)$ is an universal covering space of X, there is a path γ in $\tilde{X}$ with initial point $\tilde{x_1}$ and terminal point $\tilde{x_2}$. Using γ define $u_* : \pi_1(\tilde{X}, \tilde{x_1}) \to \pi_1(\tilde{X}, \tilde{x_2})$ to be $u_*(\alpha) = \gamma^{-1}\alpha\gamma$. Then u_* is an isomorphism and thus $p_*\pi_1(\tilde{X}, \tilde{x_1})$ and $p_*\pi_1(\tilde{X}, \tilde{x_2})$ are conjugate subgroup of $\pi_1(X, x)$. Therefore from Proposition 2.1 there exists a $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$.

□

Theorem 2.9. *Let $(\tilde{X}, p)$ be a covering space of a locally path connected space X. Then there exists a $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ iff $(\tilde{X}, p)$ is a regular covering space of X.*

Proof. Using **Proposition 2.6** and **Proposition 2.7** , it follows.

□

Theorem 2.10. *Regular covering is Galois.*

Proof. Let $(\tilde{X}, p)$ be a regular covering space of X. Then from **Theorem 2.9**, there exists a $\phi \in Deck_X(\tilde{X})$ such that $\phi(\tilde{x_1}) = \tilde{x_2}$ for $\tilde{x_1}, \tilde{x_2} \in \tilde{X}$ i.e $Deck_X(\tilde{X})$ acts transitively on the set $p^{-1}(x)$. Thus from **Definition 1.2** $(\tilde{X}, p)$ is a Galois covering of X.

□

3. STUDY OF THE GALOIS COVERING USING SHEETS OF THE COVERING

Theorem 3.1. *If $(\tilde{X}, q)$ is a p-sheeted covering space of a locally path connected space X , where p is the smallest prime dividing the order of $\pi_1(X, x)$, then $(\tilde{X}, q)$ is a Galois covering.*

Proof. Let $(\tilde{X}, q)$ is a p-sheeted covering space of a locally path connected space X , where p is the smallest prime dividing the order of $\pi_1(X, x)$. We know that the number of sheets of the covering is equal to the index of the subgroup $q_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$. Therefore $q_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$ of index p , where p is the smallest prime dividing the order of $\pi_1(X, x)$.Hence from **lemma 1.4**, $q_*\pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$. Therefore $(\tilde{X}, q)$ is a regular covering space of X and hence from **Theorem 2.10** ,$(\tilde{X}, q)$ is a Galois covering. □

Corollary 3.2. *If $(\tilde{X}, p)$ is a 2-sheeted covering space of a locally pathconnected space X,then this covering is Galois.*

Proof. Proof follows from **Theorem 3.1**.

□

Theorem 3.3. *Let $(\tilde{X}, q)$ is a covering space of a locally path connected space X such that the order of $\pi_1(X, x)$ is pn, where p is a prime, n is a positive integer and $p > n$.Let $q_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$ of order p , then the n-sheeted covering space $(\tilde{X}, q)$ is a Galois covering.*

Proof. Let $(\tilde{X}, q)$ is a covering space of a locally path connected space X such that the order of $\pi_1(X, x)$ is pn, where p is a prime , n is a positive integer and $p > n$. We know that the number of sheets of the covering is equal to the index of the subgroup $q_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$. Now Let $q_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$ of order p. So $(\tilde{X}, q)$ is a n-sheeted covering space of X. Now from **lemma 1.5** , $q_*\pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$. Therefore $(\tilde{X}, q)$ is a regular covering space of X and hence from **Theorem 2.10** , $(\tilde{X}, q)$ is a Galois covering.

□

Theorem 3.4. *Let $(\tilde{X}, q)$ is a covering space of a locally path connected space X such that the order of $\pi_1(X, x)$ is p^n, where p is a prime and n is a positive integer . If $q_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$ of order p^{n-1} , then the p-sheeted covering space $(\tilde{X}, q)$ is a Galois covering.*

Proof. Let $(\tilde{X}, q)$ is a covering space of a locally path connected space X such that the order of $\pi_1(X, x)$ is p^n, where p is a prime and n is a positive integer.Let $q_*\pi_1(\tilde{X}, \tilde{x})$ is a subgroup of $\pi_1(X, x)$ of order p^{n-1}. We know that the number of sheets of the covering is equal to the index of the subgroup $q_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$. So $\pi_1(\tilde{X}, \tilde{x})$ is a p sheeted covering space of X. Now from **lemma 1.5**, $q_*\pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$. Therefore $(\tilde{X}, q)$ is a regular covering space of X and hence from **Theorem 2.10** , $(\tilde{X}, q)$ is a Galois covering. □

4. SOME APPLICATIONS OF THE GALOIS COVERING USING SHEETS OF THE COVERING

Example 4.1. Let $p : S^2 \to RP^2$ denote the natural map of the 2-sphere onto its quotient space(by indentifying the antipodal points), the real projective plane ; then (S^2, p) is a covering space of RP^2. Because S^2 is simply connected , it is a universal covering space. So $\pi_1(S^2)$ is trivial . Now $\pi_1(RP^2)$ is of order 2. As $\pi_1(S^2)$ is trivial , so index of $p_*\pi_1(S^2)$ is the order of the group $\pi_1(RP^2)$ that is 2 and $p_*\pi_1(S^2)$ is a normal subgroup of $\pi_1(RP^2)$. So (S^2, p) is a regular covering and hence Galois covering of RP^2.

Example 4.2. For an n sheeted covering $q : S^1 \to S^1$,$z \to z^n$; the deck transformations form the group $Deck_x(\tilde{X})$ is Z_n. Now we know that for any point $x \in X$ and for any point $\tilde{x} \in q^{-1}(x)$, the group $Deck_x(\tilde{X})$ is isomorphic to $N(q_*\pi_1(\tilde{X}, \tilde{x}))/q_*\pi_1(\tilde{X}, \tilde{x})$ where $(\tilde{X}, q)$ is a covering space of X and $N(q_*\pi_1(\tilde{X}, \tilde{x}))$ is the normalizer of $q_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi(X, x)$. Now if $n = p$ is a prime then $q_*\pi_1(S^1)$ is a normal subgroup of $\pi_1(S^1)$. So (S^1, q) is a regular covering of S^1 and hence is a Galois covering of S^1.

Example 4.3. Since $\pi_1(T)$=$Z \times Z$ is abelian , where T is the torus $S^1 \times S^1$, so every subgroup of T is normal. Therefore every covering of T is regular and hence every covering of T is Galois covering.

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References

1. Massey,W.S.(1984):Algebraic topology an Introduction,Springer Verlag.
2. Spanier,E.H.(1978): Algebraic Topology, Tata McGraw-Hill Publishing Company Ltd,New Delhi.
3. Munkers,J.R.(1975); Topology a first course,Prentice Hall Inc.
4. Adhikari,M.R.;Rana,P.K. and Adhikari,A.(1999): A study of covering spaces through a functor, Review Bull Cal.Math.Soc.,7,(1 and 2),75-80.
5. Adhikari,M.R.(2016):Basic Algebraic Topology and its Applications, Springer.
6. Rana ,P.K.(2009):A study of the group of covering transformation through functors, Mathematica Bilten,33(LIX)21-25.
7. Satya Deo,*Algebraic Topology,Springer Nature Singapore Pte Ltd. and Hindustan Book Agency, 2018*

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