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ONE COMBINATORIAL PROBLEM, BUT MANY SOLUTIONS

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Abstract. The uniqueness of the solution to each task depends on a properly defined and posed problem. Simple problems can hide plenty of additional assumptions that need to resolve to arrive at a unique solution. In the case of combinatorial-type problems, the correct definition of the required outcomes is perhaps the most pronounced problem leading to many doubts and various outcomes.

1. Introduction and preliminaries

Combinatorics is a mathematical discipline that is developing rapidly. The development of the theory of partitions deserves special attention. In this paper, we will try to reduce the calculation of the number of outcomes of the division problems to the determination of the number of partitions. An example of this can be found in the works [4], [5], [6]. The reader can find a similar issue from [1], [2]. In paper [7], the authors reduced the problem of counting the number of solutions of equations to counting partitions.
We will solve the following two problems in parallel:
Problem A We have five gifts available. In how many ways can they be divided among the three children?
Problem B We have $n, (n \geq 1)$ gifts available. In how many ways can they be divided into $m, (m \geq 1)$ children?

Problem B is a general setting of Problem A. We will solve them in parallel for at least two fundamental reasons. The first is to make it easier to understand the solution of the Problem B, and the second is to check the accuracy of a simple example. The given problem A is taken from the primary school textbook and is an example of a task that will only cause more confusion for gifted students. In the first place, we want our text tasks to be as short as possible. This is precisely the most common cause of task indeterminacy. In tasks of type **B**, doubts arise, first of all, from different answers to the following questions:

(1) Are the gifts different from each other?
(2) Do we distinguish children from each other?

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(3) Can one gift be divided into more than one child? (It is not excluded that among the gifts there are soccer balls that can be divided among several children.)
(4) Do all gifts have to be given out?
(5) Does each child have to get at least one gift?

Someone may distinguish cases in which only part of the gifts or only some children differ from each other can also be in Problem B. In all cases, it is necessary to distinguish between the number of gifts and children. Consideration will also be given to problems when only a certain number of gifts are distributed, as when only some children receive gifts. In addition to these mandatory questions, additional conditions can be asked, such as:

- No child should receive more (or less) than k, $(k \geq 1)$ gifts.
- That the difference between the number of gifts of any two children should not exceed l, $(l \geq 1)$.
- One part of the gift can be shared among several children etc.

These and similar conditions change the number of possible outcomes. We will express the answers to Problem A and Problem B as *Solution x.A* and *Solution x.B* for each considered case separately. For a simpler notation of the solution, we will use the following annotations and co-factors.

2. Co-factors and notations

Any decomposition of the number n into the sum of m natural parts is called a partition of the number n of the class m. All partition of number n of class m is given with:

$$n = i_1 + i_2 + \ldots + i_m, \quad i_1 \geq i_2 \geq \cdots \geq i_m \geq 1, \tag{2.1}$$

The number of all partitions of the number n of class m will be denoted by $p_m(n)$. If some of the parts are allowed to be equal to zero, we will call such a partition extended. We will mark their number with $p_{m(+0)}(n)$. With $p_{m(i_1 \leq f)}(n)$ we will denote the number of partitions of the number n of the class m in which the first parts does not exceed the natural number f.

We will consider partition (2.1) as a sequence of numbers: $i_1, i_2, \ldots, i_m$. In last sequence there can also be groups with equal parts. Suppose that among the parts there is λ_1 are equal with i_1, λ_2 are equal with i_2, $\ldots$, λ_m are equal with i_m. To each partition (2.1) we can then associate the following five co-factors:

- **Co-factor A.** We will use it in cases where we distinguish gifts but not children. In that case, we increase each partition by an additional number of outcomes. According to the principle of product rule, we calculate the co-factor A for every partition (2.1), with the formula:

$$A(i_1 + i_2 + \ldots + i_m) = \binom{n}{i_1} \cdot \binom{n - i_1}{i_2} \cdot \ldots \cdot \binom{n - i_1 - \ldots - i_{m-1}}{i_m} \cdot \frac{1}{\lambda_1! \cdot \lambda_2! \cdot \ldots \cdot \lambda_m!}.$$

- **Co-factor B.** We use to calculate the total number of outcomes when the children differ and the gifts do not. For every partition (2.1), we calculate co-factor B with the formula:

$$B\left(i_1+i_2+\ldots+i_m\right)=\frac{m!}{\lambda_1!\cdot\lambda_2!\cdot\ldots\cdot\lambda_m!}.$$

- **Co-factor D.** We will use it when we determine the number of additional outcomes for each partition (2.1) when both children and gifts differ. First we calculate the co-factor C,

$$C\left(i_1+i_2+\ldots+i_m\right)=\binom{n}{i_1}\cdot\binom{n-i_1}{i_2}\cdot\ldots\cdot\binom{n-i_1-\ldots-i_{m-1}}{i_m},$$

then we calculate the co-factor D with the formula

$$\begin{aligned}D\left(i_1+i_2+\ldots+i_m\right)&=B\left(i_1+i_2+\ldots+i_m\right)\cdot C\left(i_1+i_2+\ldots+i_m\right)\\&=A\left(i_1+i_2+\ldots+i_m\right)\cdot m!.\end{aligned}$$

- **Co-factor E.** We will use it when determining the number of outcomes in which gifts can be repeated, each k time. We define it only for partitions of the number $k\cdot n$ of class m where the all parts cannot be greater than n. Let's replace the observed series of parts $k\cdot n=i_1+i_2+\ldots+i_m$, $n\geq i_1\geq i_2\geq\ldots\geq 1$, with a complementary series in relation to n, i.e. with $n-i_1, n-i_2,\ldots,n-i_m$. Suppose that among them there is σ_1 are equal with $n-i_1$, σ_2 are equal with $n-i_2$, …, σ_m are equal with $n-i_m$. Now we calculate the co-factor E with the formula:

$$\begin{aligned}&E\left(i_1+i_2+\ldots+i_m\right)\\&\quad=\binom{n}{n-i_1}\cdot\binom{n-i_1}{n-i_2}\cdot\ldots\cdot\binom{n-i_1-\ldots-i_{m-1}}{n-i_m}\cdot\frac{1}{\sigma_1!\cdot\sigma_2!\cdot\ldots\cdot\sigma_m!}.\end{aligned}$$

3. Possible solutions to Problems A and B

The solution to Problems A or B essentially depends on the answers to questions: 1 - 5. The "yes" and "no" answers will be split into 32 cases into eight groups with four subcases each. Let's consider each of the following 32 instances separately.

The first four cases consider number of outcomes when the children do not differ from each other, and neither do the gifts.

(3.1.) {No,No,No,Yes,Yes}.

Solution 3.1.A When the children do not differ from each other, and neither do gifts, it is identical to when we decompose the natural number five into the sum of three positive parts. In this case are: $3+1+1$ and $2+2+1$, a total of two solutions.

Solution 3.1.B By analyzing as previous case, for $m>n$, we easily conclude that there is no solution for Problem B and for $m\leq n$ the total number of solutions is $p_m(n)$.

Remark 3.1. Formulas for calculating $p_m(n)$, for $m = 1, 2, \ldots, 10$ are given in [4] and [5].

(3.2.) {No,No,No,No,Yes}.
Solution 3.2.A Since not all gifts are distributed, and each child must receive a gift, it is sufficient to separately calculate the number of possible partitions when dividing: $3, 4, 5$ gifts. We find that all possible partitions are specified with: $\underbrace{1+1+1}$, $\underbrace{2+1+1}$, $\underbrace{2+2+1, 3+1+1}$, a total of four solutions.
Solution 3.2.B Based on what has been said in previous case, for $m > n$, Problem B has no solution and for $m \leq n$ the total number of outcomes is: $p_m(m) + p_m(m+1) + \ldots + p_m(n)$.

(3.3) {No,No,No,Yes,No}.
Solution 3.3.A Since each child does not have to receive a gift, each solution case corresponds to one extended partition but no longer than three parts. The required partitions are: $5+0+0$, $4+1+0$, $3+2+0$, $3+2+1$, $2+2+1$, a total of five solutions.
Solution 3.3.B Under the given assumptions, the problem always has a solution. The resulting number is $1+p_{m(+0)}\left(n\right)$. For calculating $p_{m(+0)}\left(n\right)$ use the formula:

$$p_{m(+0)}(n) = \sum_{j=1}^{m} p_j\left(n\right) = p_m\left(n+m\right). \tag{3.1}$$

Formula (3.1) is known as a horizontal total additivity and is found in paper [4].

(3.4.) {No,No,No,No,No}.
Solution 3.4.A In addition to those we took in the previous case, we should also add all the remaining extended partitions of numbers: 5,4,3,2,1 and 0 of class three. The number of solutions is:

$$p_{3(+0)}(5) + p_{3(+0)}(4) + p_{3(+0)}(3) + p_{3(+0)}(2) + p_{3(+0)}(1) + 1.$$

By simply writing out all possible extended partitions, we find that the total number of solutions is 16.
Solution 3.4.B Under the given assumptions, problem always has a solution given by

$$\sum_{i=0}^{n} p_{m(+0)}(i) = p_{m(+0)}(n) + p_{m(+0)}(n-1) + \ldots + p_{m(+0)}(0)$$

$$= p_m(n+m) + p_m(n+m-1) + \ldots + p_m(m) = \sum_{j=m}^{n+m} p_m(n+j).$$

The following four cases determine the number of possible outcomes when the gifts differ and the children do not.

(3.5.) {Yes,No,No,Yes,Yes}.
Solution 3.5.A The case when the gifts differ from each other increases the number of possible outcomes for each partition separately. The number of

these outcomes is determined by co-factor A. Hence, each of the partitions: $3+1+1$ and $2+2+1$ should be additionally differentiated with ten and fifteen arrangements respectively. The total number of possible solutions is 25.

Solution 3.5.B By analyzing as previous case, we can easily conclude that for $m > n$, there is no solution for Problem B. For $m \leq n$ the total number of outcomes are:

$$\sum_{i_1 \geq i_2 \geq \ldots \geq i_m \geq 1}^{i_1+i_2+\cdots+i_m=n} A(i_1 + i_2 + \ldots + i_m).$$

(3.6.) {Yes,No,No,No,Yes}.

Solution 3.6.A It is necessary to assemble all the partitions of class three of the numbers: three, four and five which are: $1+1+1$, $2+1+1$, $3+1+1$ and $2+2+1$. Then multiply binomial coefficient $\binom{5}{i}$, $i = 3, 4, 5$ with the co-factor A for each partition. That's 10, 30, 10 and 15 additional outcomes respectively. The solution is the sum of all intermediate results. A total number of solutions is 65.

Solution 3.6.B By analyzing as previous case, we conclude that there for $m > n$, Problem B has no solution. For $m \leq n$ the total number of outcomes is

$$\sum_{j=m}^{n} \sum_{i_1 \geq i_2 \geq \ldots \geq i_m \geq 1}^{i_1+i_2+\ldots+i_m=j} \binom{n}{j} A(i_1 + i_2 + \ldots + i_m).$$

(3.7.) {Yes,No,No,Yes,No}.

Solution 3.7.A In this case, instead of partitions, extended partitions should be introduced. Extended partitions are: $5+0+0$, $4+1+0$, $3+2+0$, $3+1+1$, $2+2+1$. The corresponding co-factors A are respectively: $1, 5, 10, 10, 15$. There a total 41 outcomes.

Solution 3.7.B Under the given assumptions, Problem B always has a solution. The total number of outcomes is $\sum_{i_1 \geq i_2 \geq \ldots \geq i_m \geq 0}^{i_1+i_2+\ldots+i_m=n} A(i_1+i_2+\ldots+i_m)$.

(3.8.) {Yes,No,No,No,No}.

*Solution 3.8.A*Table 1. contains all possible outcomes, a total of 187 solutions.

Solution 3.8.B By analyzing as in the example given in the Table 1., we conclude that the Problem B always has a solution given by:

$$\sum_{j=0}^{n} \sum_{i_1 \geq i_2 \geq \ldots \geq i_m \geq 0}^{i_1+i_2+\ldots+i_m=j} \binom{n}{j} A\left(i_1 + i_2 + \ldots + i_m\right).$$

When we distinguish children but not a gifts, then each arrangement corresponding to one partition generates as many new outcomes as there are ways to arrange the elements of the same partition in the sum. They are a permutation with repetition of the elements that make up that partition. We increase the number of additional outcomes by co-factor B.

(3.9.) {No,Yes,No,Yes,Yes}.

Solution 3.9.A As the children differ from each other, partition parts

TABLE 1.

i	$Partitions$	$Co-factor A$	$\binom{5}{i}$	$Total$
0	$0+0+0$	1	1	1
1	$1+0+0$	1	5	5
2	2+0+0	1	10	10
	1+1+0	1	10	10
3	3+0+0	1	10	10
	2+1+0	3	10	30
	1+1+1	1	10	10
4	4+0+0	1	5	5
	3+1+0	4	5	20
	2+2+0	3	5	15
	2+1+1	6	5	30
5	5+0+0	1	1	1
	4+1+0	5	1	5
	3+2+0	10	1	10
	3+1+1	10	1	10
	2+2+1	15	1	15

should be ruled out by the law of commutativity. There are only two partitions that satisfy the conditions $3+1+1$ and $2+2+1$. We easily find that: $B(3+1+1) = 3$ and $B(2+2+1) = 3$, so the total number of possible outcomes is six.

Solution 3.9.B By analyzing as previous case, we can easily conclude that for $m > n$, there is no solution for Problem B, and for $m \le n$, a total number of solutions is: $\sum_{i_1 \ge i_2 \ge \ldots \ge i_m \ge 1}^{i_1+i_2+\ldots+i_m=n} B\left(i_1+i_2+\ldots+i_m\right).$

(3.10.) {No,Yes,No,No,Yes}.

Solution 3.10.A We will distinguish the cases when distributing three, four and five gifts. All possible partitions are:$1+1+1$, $2+1+1$, $3+1+1$, $2+2+1$. Corresponding co-factor B are respectively: $1, 3, 3, 3$. A total ten outcomes.

Solution 3.10.B By analysis as previous case, we conclude that for $m > n$, there is no solution for task, and for $m \le n$, the total number of solutions is: $\sum_{j=m}^{n} \sum_{i_1 \ge i_2 \ge \ldots \ge i_m \ge 1}^{i_1+i_2+\ldots+i_m=j} B\left(i_1+i_2+\ldots+i_m\right).$

(3.11.) {No,Yes,No,No,No}.

Solution 3.11.A Extended partitions should be considered instead of partitions. Table 2. contains all possible outcomes, a total of 56 solutions.

Solution 3.11.B By analyzing as previous case, we conclude that the

TABLE 2.

i	$Partitions$	$Co-factor B$	$Total$
0	$0+0+0$	1	1
1	$1+0+0$	3	3
2	2+0+0	3	6
	1+1+0	3	
3	3+0+0	3	10
	2+1+0	6	
	1+1+1	1	
4	4+0+0	3	15
	3+1+0	6	
	2+2+0	3	
	2+1+1	3	
5	5+0+0	3	21
	4+1+0	6	
	3+2+0	6	
	3+1+1	3	
	2+2+1	3	

Problem B always has a solution given by:

$$\sum_{j=0}^{n} \sum_{i_1 \geq i_2 \geq \ldots \geq i_m \geq 0}^{i_1+i_2+\ldots+i_m=j} B\left(i_1+i_2+\ldots+i_m\right) = \binom{n+m}{m}. \tag{3.2}$$

Another way. If we denote number the gifts with the: $0, 1, 2, \ldots, m$ where we use zero when we do not assign a gift to some of the positions, then the number of required outcomes is the number of all strings of length n with the $m+1$ given elements. The total number of different strings given by $\binom{m+1+n-1}{n} = \binom{m+n}{n}$. The result obtained is the sum of (3.2) given in Solution 1.B and represents the result more.

(3.12.) {No,Yes,No,Yes,No}.

Solution 3.12.A All possible partitions are: $5+0+0$, $4+1+0$, $3+2+0$, $3+1+1$ and $2+2+1$. Co-factors B respectively are: $3, 6, 6, 3, 3$. A total of 21 outcomes.

Solution 3.12.B Problem B always has a solution, a total number of outcomes are:

$$B\left(p_{m(+0)}(n)\right) = \sum_{i_1 \geq i_2 \geq \ldots \geq i_m \geq 0}^{i_1+i_2+\ldots+i_m=n} B\left(i_1+i_2+\ldots+i_m\right) \tag{3.3}$$

Another way. The required number of outcomes is the difference between the number of outcomes with $m+1$ elements and with m elements. So, the required number og outcomes is: $\binom{m+n}{n} - \binom{m+n-1}{m}$, which is also the sum of (3.3). The result obtained represents the result more.

Now suppose that both gifts and children differ. In that case, we use the co-factor D to calculate the additional number of outcomes per each partition.

(3.13.) {Yes,Yes,No,Yes,Yes}.

Solution 3.13.A There are only two possible partitions: $3+1+1$ and $2+2+1$. Co-factors A are: 10 and 15. Each partition is increased by 3! of outcomes. A total number of outcomes are $(10+15)\cdot 3! = 150$.

Solution 3.13.B Based on the above for $m > n$, there is no solution for Problem B, and for $m \le n$, the total number of solutions is:

$$m! \cdot \sum_{i_1 \ge i_2 \ge \ldots \ge i_m \ge 1}^{i_1+i_2+\ldots+i_m=m} A\left(i_1 + i_2 + \ldots + i_m\right)$$

(3.14.) {Yes,Yes,No,No,Yes}.

Solution 3.14.A Table 3. contains all possible outcomes, a total of $3!\cdot 65 = 390$ solutions.

Solution 3.14.B Based on the above for $m > n$, there is no solution for

Table 3.

i	*Partitions*	$\binom{5}{i}$	A	*Total*
3	$1+1+1$	10	1	10
4	$2+1+1$	5	6	30
5	$3+1+1$	1	10	10
	$2+2+1$	1	15	15

Problem B, and for $m \le n$, the total number of solutions is:

$$m! \cdot \sum_{j=m}^{n} \sum_{i_1 \ge i_2 \ge \ldots \ge i_m \ge 1}^{i_1+i_2+\ldots+i_m=m} \binom{n}{j} \cdot A\left(i_1 + i_2 + \ldots + i_m\right).$$

(3.15.) {Yes,Yes,No,Yes,No}.

Solution 3.15.A Extended partitions are: $5+0+0$, $4+1+0$, $3+2+0$, $3+1+1$ and $2+2+1$. Co-factors D are: $3, 30, 60, 60, 90$. A total number of outcomes are 243.

Solution 3.15.B Problem B always has a solution, a total number of outcomes are:

$$\sum_{j=1}^{m} \sum_{i_1 \ge i_2 \ge \ldots \ge i_j \ge 0}^{i_1+i_2+\ldots+i_j=n} D\left(i_1 + i_2 + \ldots + i_j\right). \tag{3.4}$$

Another way. The solution is each arrangement of a single array of length m in which each member can be from a set of n different objects. There are m^n such sequences. The obtained result is also the required sum in (3.4).

Third way. To count the number of all outcomes, let's see one partition of number n class $j, (m \le j \le n)$, $n = i_1 + i_2 + \ldots + i_j$, $i_1 \ge i_2 \ge \ldots \ge i_j \ge 1$. Let's note that in this case $\binom{n}{i_1}$ of choice can be formed for the first part i_1. Reasoning similarly for the others, according to the principle of combining, we conclude that the total corresponds to that specific arrangement $\binom{n}{i_1}\binom{n-i_1}{i_2}\ldots\binom{n-i_1-\ldots-i_{j-1}}{i_j}$, different outcomes with the observed number of gifts per child. It is enough to add all those numbers by all partitions. Thus, the solution is given by the formula:

$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{n-i_1}\cdots\sum_{j_m=0}^{n-i_1-\ldots-i_m}\binom{n}{j_1}\binom{n-i_1}{j_2}\ldots\binom{n-i_1-\ldots-i_{m-1}}{j_m}.$$

At first glance, the second and third solutions are not the same, however, it can be easily proved that the sum specified in the third solution is exactly equal to m^n. In this sense, you should know that the sum of binomial coefficients of order n is equal to 2^n. Also, it can be seen that addition can be done separately for each sum starting from right to left. The index i_m by which the first addition will start appears only in the last binomial coefficient, the other binomial coefficients are constant for that addition and as such do not affect the sum. After addition by i_m, addition by index i_{m-1} continues in a similar way, etc. Hence,

$$\sum_{i_m=0}^{n}\binom{n-i_1-\ldots-i_{m-1}}{i_m} = 2^{n-i_1-\ldots-i_{m-1}}$$

$$\sum_{i_{m-1}=0}^{n-i_1-\ldots-i_{m-2}}\binom{n-i_1-\ldots-i_{m-2}}{i_{m-1}}2^{i_{m-1}} = 3^{n-i_1-\ldots-i_{m-2}}$$

...

$$\sum_{i_1=0}^{n}\binom{n}{i_1}(m-1)^{n-i_1} = m^n,$$

By mathematical induction it can be proved that the end result is m^n.

(3.16.) {Yes,Yes,No,No,No}.

Solution 3.16.A Table 4. contains all possible outcomes, a total of 1024 solutions.

Solution 3.16.B Problem B always has a solution, a total number of outcomes are:

$$\sum_{j=0}^{n}\binom{n}{j}\sum_{i_1\ge i_2\ge\ldots\ge i_m\ge 0}^{i_1+i_2+\cdots+i_m=n} D\left(i_1+i_2+\ldots+i_m\right).$$

Assume that each gift can be given to several children at the same time. For simplicity, we will assume that each gift is divisible on exactly $k, (k \ge 1)$, children. Also, in the following text, we will consider that in

TABLE 4.

	Partitions	D	*Total*
0	$0+0+0$	1	1
1	$1+0+0$	15	15
2	2+0+0	30	90
	1+1+0	60	
3	3+0+0	30	270
	2+1+0	180	
	1+1+1	60	
4	4+0+0	15	405
	3+1+0	120	
	2+2+0	90	
	2+1+1	180	
5	5+0+0	3	243
	4+1+0	30	
	3+2+0	60	
	3+1+1	60	
	2+2+1	90	

Problem A it is implied that $k = 2$.
For $k = 1$, the task is reduced to already solved cases given under 3.1. - 3.16.
For $k \geq m$ it is easy to see that there is only one possible arrangement because if a gift is shared it will go to everyone. Therefore, we will continue to consider only case $1 < k < m$.

(3.17.) {No,No,Yes,Yes,Yes}.
Solution 3.17.A It is necessary to observe all partitions of the number ten of class three in which the first part is less than or equal to five. This is because the maximum number of gifts that a child can receive remains equal to the total number of gifts. All possible partitions are: $5 + 4 + 1$, $5 + 3 + 2$, $4 + 4 + 2$, $4 + 3 + 3$. The total number of solutions is four.

Remark 3.2. If five gifts are distributed, not one child can get six. The possibility of multiple sharing of gifts can increase the number of gifts only up to the total number of gifts.

Solution 3.17.B As one gift is divided among k children, the possible number of outcomes are all partitions of the number $k \cdot n$ of the class m whose first part does not exceed the number of gifts n. Based on the previous argumentation, we conclude that the Problem B for $k \cdot n < m$, has no solution and for $k \cdot n \geq m$, the total number of solutions is $p_{m(i_1 \leq n)}(k \cdot n)$.

(3.18.) {No,No,Yes,No,Yes}.
Solution 3.18.A The smallest number of gifts that covers all children is two. The total number of outcomes consists of all outcomes for the distribution of 2, 3, 4 and 5 gifts. Table 5. contains all possible outcomes, a total of 10 solutions.

Solution 3.18.B The smallest number of gifts that can be distributed so that each child receives at least one gift, with one gift going to k children, is $\left[\frac{m}{k}\right]+1$. In this way, we conclude for $k \cdot n < m$, that there is no solution for Problem B and for $k \cdot n \geq m$ the total number of solutions is given by the formula: $\sum_{j=\left[\frac{m}{k}\right]+1}^{m} p_{m(i_1 \leq j)}(j \cdot n)$.

Table 5.

5	4	3	2
$5+4+1$	$4+3+1$	$3+2+1$	$2+1+1$
$5+3+2$	$4+2+2$	$2+2+2$	
$4+4+2$	$3+3+2$		
$4+3+3$			

(3.19.) {No,No,Yes,Yes,No}.
Solution 3.19.A All possible outcomes are given with the following partitions: $5+5+0$, $5+4+1$, $5+3+2$, $4+4+2$, $4+3+3$, a total of five solutions.

Solution 3.19.B Problem B always has a solution, a total number of outcomes are: $p_{m(i_1 \leq n, +0)}(k \cdot n)$.

(3.20.) {No,No,Yes,No,No}.
Solution 3.20.A Possible outcomes are all extended partitions whose first part is not greater than the number of gifts being shared. Table 6. contains all possible outcomes, a total of 16 solutions.

Solution 3.20.B The task under the given assumptions always has a solution where the total number of solutions is calculated according to the formula: $\sum_{j=0}^{n} p_{m(i_1 \leq j, +0)}(k \cdot n)$.

Now let's assume that children are different and gifts are not. In that case, apart from the number of gifts, we differentiate the outcomes according to the mutual arrangement among the children. We have already emphasized that in that case the total number of outcomes for each partition should be increased by the co-factor B.

TABLE 6.

0	1	2	3	4	5
$0+0+0$	$1+1+0$	$2+2+0$	$3+3+0$	$4+4+0$	$5+5+0$
		$2+1+1$	$3+2+1$	$4+3+1$	$5+4+1$
			$2+2+2$	$4+2+2$	$5+3+2$
				$3+3+2$	$4+4+2$
					$4+3+3$

(3.21.) {No,Yes,Yes,Yes,Yes}.
Solution 3.21.A Possible partitions are: $5+4+1$, $5+3+2$, $4+4+2$, $4+3+3$. Responding co-factors B are: $6, 6, 3, 3$. A total number of outcomes are 18.
Solution 3.21.B In this way, we conclude for $k \cdot n < m$, that there is no solution for Problem B and for $k \cdot n \geq m$ the total number of solutions is given by the formula: $\sum_{n \geq i_1 \geq \ldots i_m \geq 1}^{i_1+i_2+\ldots+i_m=k\cdot n} B\left(p_{m(i_1 \leq n)}(k \cdot n)\right)$.
(3.22.) {No,Yes,Yes,No,Yes}.
Solution 3.22.A Table 7. contains all possible outcomes, a total of 40 solutions.
Solution 3.22.B In this way, we conclude for $k \cdot n < m$, that there is no

TABLE 7.

2	B	3	B	4	B	5	B
$2+1+1$	3	$3+2+1$	6	$4+3+1$	6	$5+4+1$	6
		$2+2+2$	1	$4+2+2$	3	$5+3+2$	6
				$3+3+2$	3	$4+4+2$	3
						$4+3+3$	3

solution for Problem B and for $k \cdot n \geq m$ the total number of solutions is given by the formula: $\sum_{j=\left[\frac{m}{k}\right]+1}^{n} B\left(p_{m(i_1 \leq j)}(k \cdot n)\right)$.
(3.23.) {No,Yes,Yes,Yes,No}.
Solution 3.23.A Table 8. contains all possible outcomes, a total of 21 solutions.
Solution 3.23.B The task always has at least one possible solution and the total number of solutions is given by the formula:

$$\sum_{j=0}^{m} \sum_{n \geq i_1 \geq \ldots \geq i_j \geq 1}^{i_1+i_2+\cdots+i_j=k\cdot n} B\left(p_{m(i_1 \leq j,+0)}(k \cdot n)\right).$$

TABLE 8.

$Partitions$	$Co-factorB$
$5+5+0$	3
$5+4+1$	6
$5+3+2$	6
$4+4+2$	3
$4+3+3$	3

Another way. The sum obtained does not differ numerically from the result obtained in (3.12.) and amounts $\binom{n+m}{n}-\binom{n+m-1}{n}$.

(3.24.) {No,Yes,Yes,No,No}.

Solution 3.24.A Table 9. contains all possible outcomes, a total of 56 solutions.

Solution 3.24.B According to the obtained result, there is always at least one possible solution, and the total number is given by the formula: $\sum_{j=0}^{n} B\left(p_{m(i_1\leq j,+0)}\left(j\cdot n\right)\right)$.

Another way. If we number the gifts with the numbers $0,1,2,\ldots$, and count the number of all different strings of length $k\cdot n$, where we use zero to mark the place where we do not distribute the gift, then the number of those strings is exactly equal to the number of strings of length n because all the elements are repeated exactly k times. Hence it follows that the number of all those sequences does not depend on k and is the same as in case (3.11.), $\binom{n+m}{n}$. The obtained result represents a higher result because it follows that the number of all partitions of the class k number n is equal to the number of those extended partitions of the class m number $k\cdot n$ whose first sum is not greater than n. ($p_m(n)=p_{m(i_1\leq n,+0)}\left(k\cdot n\right)$).

Suppose now that the gifts are different and the children are not. The additional number of solutions per each partition is determined by co-factors E.

(3.25.) {Yes,No,Yes,Yes,Yes}.

Solution 3.25.A Table 10. contains all possible outcomes, a total of 40 solutions.

Solution 3.25.B In this way, we conclude for $k\cdot n<m$, that there is no solution for Problem B and for $k\cdot n\geq m$ the total number of solutions is given by the formula: $\sum_{n\geq i_1\geq i_2\geq\ldots\geq i_m\geq 1}^{i_1+i_2+\ldots+i_m=k\cdot n} E\left(p_{m(i_1\leq j,+0)}\left(k\cdot n\right)\right)$.

(3.26.) {Yes,No,Yes,No,Yes}.

Solution 3.26.A Table 11. contains all possible outcomes, a total of 58 solutions.

Solution 3.26.B In this way, we conclude for $k\cdot n<m$, that there is no solution for Problem B and for $k\cdot n\geq m$ the total number of solutions is given by the formula: $\sum_{j=\left[\frac{m}{k}\right]+1}^{n} E\left(p_{m(i_1\leq j,+0)}\left(j\cdot n\right)\right)$.

Table 9.

i	$Partitions$	$Co-factorB$
0	$0+0+0$	1
1	$1+1+0$	3
2	2+2+0	3
	2+1+1	3
3	3+3+0	3
	3+2+1	6
	2+2+2	1
4	4+4+0	3
	4+3+1	6
	4+2+2	3
	3+3+2	3
5	5+5+0	3
	5+4+1	6
	5+3+2	6
	4+4+2	3
	4+3+3	3

Table 10.

$Parttitions$	$Complements$	$Co-factorE$
$5+4+1$	$0,1,4$	5
$5+3+2$	$0,2,3$	10
$4+4+2$	$1,1,3$	10
$4+3+3$	$1,2,2$	15

(3.27.) {Yes,No,Yes,Yes,No}.

Solution 3.27.A Table 12. contains all possible outcomes, a total of 41 solutions.

Solution 3.27.B With a similar reasoning as in the example, while recognizing that the number of outcomes is additionally increased by the distribution of a smaller number of gifts than possible because they differ

TABLE 11.

i	$Partitions$	$Complement$	$Co-factorE$	$Total$
2	$2+1+1$	$0,1,1$	1	1
3	3+2+1	0,1,2	3	4
	2+2+2	1,1,1	1	
4	4+3+1	0,1,3	4	13
	4+2+2	0,2,2	3	
	3+3+2	1,1,2	6	
5	5+4+1	0,1,4	5	40
	5+3+2	0,2,3	10	
	4+4+2	1,1,3	10	
	4+3+3	1,2,2	15	

TABLE 12.

$Partition$	$Coplement$	$Co-factorE$
$5+5+0$	$0,0,5$	1
$5+4+1$	$0,1,4$	5
$5+3+2$	$0,2,3$	10
$4+4+2$	$1,1,3$	10
$4+3+3$	$1,2,2$	15

from each other. Problem B always has a solution, a total number of outcomes are: $\sum_{n\geq i_1\geq\ldots\geq i_m\geq 0}^{i_1+i_2+\ldots+i_m=k\cdot n} E\left(p_{m(i_1\leq n,+0)}\left(k\cdot n\right)\right)$.

(3.28.) {Yes,No,Yes,No,No}.

Solution 3.28.A Table 13. contains all possible outcomes, a total of 69 solutions.

Solution 3.28.B We conclude for $k\cdot n<m$, that there is no solution for Problem B and for $k\cdot n\geq m$ the total number of solutions is given by the formula: $\sum_{i=0}^{n}\sum_{i\geq j_1\geq\ldots\geq j_i\geq 0}^{j_1+j_2+\ldots+j_i=k\cdot i} E\left(p_{m(j_1\leq i,+0)}\left(i\cdot n\right)\right)$.

There is still a case when we distinguish between children and gifts. In this case, we increase each possible partition additionally by the product of co-factors B and E.

(3.29.) {Yes,Yes,Yes,Yes,Yes}.

Solution 3.29.A Table 14. contains all possible outcomes, a total of 240 solutions.

TABLE 13.

i	$Parttions$	$Complement$	$Co-factorE$	$Total$
0	$0+0+0$	$0,0,0$	1	1
1	$1+1+0$	$0,0,1$	1	1
2	2+2+0	0,0,2	2	4
	2+1+1	0,1,1	2	
3	3+3+0	0,0,3	1	5
	3+2+1	0,1,2	3	
	2+2+2	1,1,1	1	
4	4+4+0	0,0,4	1	17
	4+3+1	0,1,3	4	
	4+2+2	0,2,2	6	
	3+3+2	1,1,2	6	
5	5+5+0	0,0,5	1	41
	5+4+1	0,1,4	5	
	5+3+2	0,2,3	10	
	4+4+2	1,1,3	10	
	4+3+3	1,2,2	15	

Solution 3.29.B We conclude for $k \cdot n < m$, that there is no solution for

TABLE 14.

$Partitions$	$Co-factorB$	$Co-factorE$	$Total$
$5+4+1$	6	5	30
$5+3+2$	6	10	60
$4+4+2$	3	20	60
$4+3+3$	3	30	90

Problem B and for $k \cdot n \geq m$ the total number of solutions is given by the formula:

$$\sum_{n\geq i_1\geq i_2\geq \ldots i_m\geq 1}^{i_1+i_2+\ldots+i_m=k\cdot n} B\left(p_{m(i_1\leq n,+0)}\left(k\cdot n\right)\right)\cdot E\left(p_{m(i_1\leq n,+0)}\left(k\cdot n\right)\right).$$

(3.30.) {Yes,Yes,Yes,No,Yes}.
Solution 3.30.A Table 15. contains all possible outcomes, a total of 250 solutions.
Solution 3.30.B We conclude for $k \cdot n < m$, that there is no solution for

TABLE 15.

2	B	E	3	B	E	4	B	E	5	B	E
$2+1+1$	3	2	3+2+1 2+2+2	6 1	3 1	4+3+1 4+2+2 3+3+2	6 3 3	4 6 6	5+4+1 5+3+2 4+4+2 4+3+3	6 6 3 3	5 10 10 15
$Total$		6			19			60			165

Problem B and for $k \cdot n \geq m$ the total number of solutions is given by the formula:

$$\sum_{i=\frac{k}{m}+1}^{n} \sum_{i \geq i_1 \geq i_2 \geq \cdots \geq 1}^{i_1+i_2+\ldots+i_m=i \cdot n} B\left(p_{m(i_1 \leq i,+0)}\left(i \cdot n\right)\right) \cdot E\left(p_{m(i_1 \leq i,+0)}\left(i \cdot n\right)\right).$$

Remark 3.3. The result is more easily obtained by comparing it with the already considered case given under (3.14.).

(3.31.) {Yes,Yes,Yes,Yes,No}
Solution 3.31.A Table 16. contains all possible outcomes, a total of 123 solutions.
Solution 3.31.B Problem B always has a solution, a total number of

TABLE 16.

$Partitions$	$Co-factorB$	$Co-factorE$	$Total$
$5+5+0$	3	1	3
$5+4+1$	6	5	30
$5+3+2$	6	10	60
$4+4+2$	3	10	30

outcomes are:

$$\sum_{n \geq i_1 \geq \ldots \geq i_m \geq 0}^{i_1+i_2+\ldots+i_m=k \cdot n} B\left(p_{m(i_1 \leq n)}\left(k \cdot n\right)\right) \cdot E\left(p_{m(i_1 \leq n)}\left(k \cdot n\right)\right).$$

(3.32.) {Yes,Yes,Yes,No,No}.
Solution 3.32.A It is necessary to sort the extended unordered partitions

for the numbers 0 to 5. Using the cofactors B and E, we get 248 outcomes.
Solution 3.32.B We conclude for $k \cdot n < m$, that there is no solution for Problem B and for $k \cdot n \geq m$ the total number of solutions is given by the formula:

$$\sum_{j=\frac{k}{m}+1}^{n} \sum_{n \geq i_1 \geq \ldots \geq i_j \geq 0}^{i_1+i_2+\ldots+i_j=k\cdot n} B\left(p_{m(i_1 \leq j)}\left(k \cdot n\right)\right) \cdot E\left(p_{m(i_1 \leq j)}\left(k \cdot n\right)\right).$$

Remark 3.4. Here also a result can be obtained more by comparing the obtained solution with the case under (3.16.). It can be concluded that the number of possible outcomes under the given conditions does not depend on the number k.

4. Conclusion

We have presented how determining the number of partitions can be used to determine the number of divisions. Apart from the standard observation of the term partition (as in (2.1)), it is necessary to introduce three more terms.
1. Extended partitions, when a zero can appear as a part.
2. Unordered partitions, when the law of commutation is cancelled.
3. Extended unordered partitions.

Knowing how to calculate the listed four types of partitions, it is possible to solve all division problems. In several cases, for calculating the total number of outcomes, some simple formulas are given. For many remaining, the task remains to simplify.

The mentioned ambiguities are not the only ones. Further elaboration on this topic deserves attention. Furthermore, we analyzed to what extent linguistic doubts can affect the solving process.

5. Declaration

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