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ON COMMON SOLUTION OF FIXED POINT PROBLEM AND CONVEX MINIMIZATION PROBLEM THROUGH MODIFIED PROXIMAL POINT ALGORITHM

KIRAN DEWANGAN*

Abstract. The study of finding common elements from the set of fixed point and set of solutions of convex minimization problems for nonlinear operators increase rapidly due to its applications in different branches of mathematics and engineering. In optimization theory, the proximal point algorithm plays an important role in solving problems with nonlinear operators. Several modifications of proximal point algorithm are introduced. Here a modified proximal point algorithm (MPPA) is introduced with strong convergence of MPPA in Hilbert spaces.

1. Introduction

Fixed point theory has a great implications in the field of analysis. Banach contraction principle [1] is widely used in existence and uniqueness theories. On the other hand to study fixed point of nonexpansive type mappings, some geometric assumptions are needed either on the structure of mappings or spaces. Determining the fixed point of nonexpansive mappings is frequent problem in Mathematics and Physical sciences.

There are several applications of fixed point theory in integral equations, differential equations, optimization theory etc. There are lots of activity in the field of optimization theory, especially study of convex minimization problem along with fixed point problem for nonlinear operators and due to this activities, researchers attracts to obtain more and more results in this field.

In the optimization theory, we often come to the problem to find a point $x \in M$ such that

$$g(x) = \min_{u \in K} g(u),$$

where $g : M \to (-\infty, \infty)$ be a proper convex and lower semi-continuous function and B be a non-empty closed convex subset of a Hilbert space M with inner

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* Corresponding author.
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product $(.,.)$ and norm $||.||$. Recall that the set of all minimizers of g on B is denoted by $argmin_{u\in B}g(u)$.

To solve optimization problems, an important tool, namely proximal point algorithm is used. Initially the proximal point algorithm (PPA) was introduced and studied by Martinet in 1970 [11]. Note that a constrained minimization problem

$$\min\{g(x) : x \in B\}$$

is called consistent, if constrained minimization problem has a solution. In 1976, Rockafellar [16] studied convergence of proximal point algorithm for finding solution of constrained minimization problem in the framework of Hilbert space as follows:

$$\begin{cases} x_1 \in M, \\ x_{k+1} = argmin_{u\in M}[g(u) + \frac{1}{2\lambda_k}||x_k - u||^2], \end{cases} \tag{1.1}$$

where $\lambda_k > 0$ for all $k \geq 1$. Rockafellar [16] proved that the sequence $\{x_k\}$ generated by (1.1) converges weakly to a minimizers of g provided that $\sum_{k=0}^{\infty} \lambda_k = \infty$.

In general, the proximal point algorithm does not necessarily converge strongly. Several researchers proposed the modification of Rockafellar PPA to have strong convergence like-in 2000, Takahashi and Kamimura [18] combined proximal point algorithm with Halpern iteration process [8] to obtain strong convergence of the sequence. In 1991, Guler [7] proposed a proper lower-semi continuous and convex function in l_2 space for which PPA converges weakly but not strongly. This arises an open question that the sequence $\{x_k\}$ converges strongly? Can PPA be modified to guarantee strong convergence? To solve these questions, several works have been done (see Reich [15], Chidume [2] etc.).

In 2003, Xu [21] proposed the following iteration scheme:

$$\begin{cases} x_0 \in M, \\ x_{k+1} = \alpha_k b + (I - \alpha_k A)Ux_k, \ k \geq 0. \end{cases} \tag{1.2}$$

Xu [21] proved that the iteration scheme (1.2) converges strongly to the unique solution of the minimization problem, where U is nonexpansive mapping in Hilbert space M and A is strongly positive bounded linear operator.

In 2006, Marino and Xu [10] extended the results of Xu [21] and Moudafi's [14] by following iteration scheme:

$$\begin{cases} x_0 \in M, \\ x_{k+1} = \alpha_k \gamma f(x_k) + (I - \alpha_k A)Ux_k, \ k \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$, A is bounded linear operator on M and U is nonexpansive mapping. They proved that the iteration scheme (1.3) converges strongly to a fixed point of U under suitable conditions, which is also a unique solution of the VIP:

$$(Ax^* - \gamma f(x^*), x^* - p) \leq 0, \ \forall\, p \in F(U).$$

In 2016, Chang [3] et al., introduced the following proximal point algorithm:

$$\begin{cases} x_1 \in B, \\ u_k = argmin_{y \in M}[g(y) + \frac{1}{2\lambda_k}||x_k - y||^2], \\ z_k = (1-\beta_k)x_k + \beta_k w_k, \ w_k \in Uu_k, \\ x_{k+1} = (1-\alpha_k)x_k + \alpha_k v_k, \ v_k \in Uz_k, \end{cases} \tag{1.4}$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$ and $\{\beta_k\}$ is a sequence in $[0,1]$ such that $0 < a < \alpha_k, \ \beta_k \leq b < 1$ for all $k \geq 1$, and $\{\lambda_k\}$ is a sequence such that $\lambda_k \geq \lambda > 0$ for all $k \geq 1$ and for some λ. They proved that the iteration scheme (1.4) converges weakly to an element of Ω, where $\Omega = F(U) \cap argmin_{u\in B} g(u) \neq \emptyset$.

In 2017, Suparatulatorn [17] et al., introduced modified proximal point algorithm combined with Halpern iteration scheme for nonexpansive mapping in complete CAT(0) space as follows:
Let g be a proper convex and lower semi-continuous function on X. The MPPA is defined by $u, x_1 \in X$ and

$$\begin{cases} y_k = argmin[g(y) + \frac{1}{2\lambda_k} d(y, x_k)^2], \\ x_{k+1} = \alpha_k u \oplus (1-\alpha_k)Uy_k, \ \forall\, k \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\lambda_k > 0$ for all $k \in \mathbb{N}$. They proved that if g has a minimizer and $\{\lambda_k\}$, $\{\alpha_k\}$ satisfy some mild conditions, then the iteration scheme (1.5) converges strongly to its minimizer.

Motivated by above work, now we introduced the following MPPA in the framework of Hilbert space. Let B be a non-empty closed convex subset of real Hilbert space M. Let $g : B \to (-\infty, \infty)$ be a proper convex and lower semi-continuous function and $U : B \to B$ is a nonexpansive mapping. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $\{x_k\}$ be a sequence defined by

$$\begin{cases} x_0 \in B, \\ z_k = argmin_{u \in B}[g(u) + \frac{1}{2\lambda_k}||u - x_k||^2], \\ w_k = Uz_k, \\ v_k = (1-\beta_k)x_k + \beta_k w_k, \\ x_{k+1} = P_K(\alpha_k \gamma f(x_k) + (I - \eta\alpha_k A))v_k, \end{cases} \tag{1.6}$$

where $\{\alpha_k\}, \{\beta_k\}$ are sequence in $[0,1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \inf \beta_k(1-\beta_k) > 0$ and $\{\lambda_k\}$ is a sequence such that $0 < \lambda \leq \lambda_k$ for all $k \geq 1$ and for some λ and $0 < \eta < \frac{2k}{L^2}$.

Our aim in this paper is to study strong convergence of the iteration scheme defined by (1.6) for nonexpansive mappings and finite family of nonexpansive mappings under some mild conditions for fixed point problems as well as convex minimization problems in Hilbert spaces which are solutions of some variational inequality problems (VIP). Our results generalizes several results.

2. Preliminaries

This section contains some basic informations and included some useful results.

Definition 2.1. [24] A function $g : B \to (-\infty, \infty)$ defined on a non-empty convex subset of a Hilbert space M is lower semi-continuous at a point $x \in B$, if

$$g(x) \leq \liminf_{k\to\infty} g(x_k),$$

for each sequence $\{x_k\}$ such that $\lim_{k\to\infty} x_k = x$. A function g is said to be lower semi-continuous on B, if it is lower semi-continuous at any point in B.

Definition 2.2. [23] A mapping $U : B \to B$ is called-

(i) nonexpansive, if

$$||Ux - Uy|| \leq ||x - y||, \ \forall \ x, y \in B,$$

(ii) $k-$ strictly pseudo-contractive, if for $k \in [0, 1]$,

$$||Ux - Uy||^2 \leq ||x - y||^2 + k||(I - U)x - (I - U)y||^2 \ \forall \ x, y \in B. \tag{2.1}$$

Remark 2.3. (i) Every nonexpansive mapping is strictly pseudocontractive.

(ii) In a real Hilbert space M, (2.1) is equivalent to

$$(Ux - Uy, x - y) \leq ||x - y||^2 - \frac{1-k}{2}||(I - U)x - (I - U)y||^2. \tag{2.2}$$

Definition 2.4. [23] For any $x \in M$, there exists a unique point $P_K x \in B$ such that

$$||x - P_K x|| \leq ||x - y||, \ \forall \ y \in B.$$

Here P_K is called metric projection of M onto B.

Remark 2.5. The metric projection P_K is a nonexpansive mapping of M onto B.

Recall that the metric projection P_K satisfies

$$(x - y, P_K x - P_K y) \geq ||P_K x - P_K y||^2, \ \forall \ x, y \in M. \tag{2.3}$$

Furthermore $P_K x$ is characterized by the property

$$(x - P_K x, y - P_K x) \leq 0, \ \forall \ y \in B. \tag{2.4}$$

Definition 2.6. [13] A nonlinear operator $U : M \to M$ is said to be

(i) monotone if

$$(x - y, Ux - Uy) \geq 0, \ \forall \ x, y \in M,$$

(ii) $\eta-$ strongly monotone, if there exists $\eta > 0$ such that

$$(x - y, Ux - Uy) \geq \eta||x - y||^2, \ \forall \ x, y \in M,$$

Definition 2.7. [23] Let M be a real Hilbert space and $U : M \to M$ be a mapping. $I - U$ is said to be demiclosed at 0 if for any sequence $\{x_k\}$ in M such that $\{x_k\}$ converges weakly to $x \in M$, and $\{Ux_k\}$ converges strongly to some y, then $Ux = y$.

Lemma 2.8. [5] *Let M be a real Hilbert space. Then for every $x, y \in M$, and every $\lambda \in [0, 1]$, the following hold:*

$$||x + y||^2 \leq ||x||^2 + 2(y, x + y),$$
$$||\lambda x + (1 - \lambda)y||^2 = \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)||x - y||^2.$$

Lemma 2.9. [20] *Assume that $\{a_k\}$ is a sequence of non-negative real numbers such that $a_{k+1} \leq (1 - \alpha_k)a_k + \sigma_k$ for all $k \geq 0$, where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\sigma_k\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\sum_{k=0}^{\infty} \alpha_k = \infty$,
(b) $\limsup_{k\to\infty} \frac{\sigma_k}{\alpha_k} \leq 0$ *or* $\sum_{k=0}^{\infty} |\alpha_k| < \infty$.

Then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.10. [19] *Let M be a real Hilbert space. Let B be a non-empty closed convex subset of M and $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta(k - \frac{L^2\eta}{2})$. Then for each $t \in (0, \min(1, \frac{1}{\tau}))$, we have*

$$||(I - t\eta A)x - (I - t\eta A)y|| \leq (1 - t\tau)||x - y||, \; x, y \in B.$$

Let B be a non-empty closed convex subset of real Hilbert space M. Let $g : B \to (-\infty, \infty)$ is proper convex and lower semi continuous. For every $\lambda > 0$, the Moreau-Yosida resolvent of g denoted by J_λ^g is defined by

$$J_\lambda^g x = argmin_{u\in B}[g(u) + \frac{1}{2\lambda_k}||x - u||^2],$$

for all $x \in M$. In [7], it was shown that the set of fixed points of the resolvent associated to g coincides with the set of minimizers of g. Also the resolvent J_λ^g of g is nonexpansive for all $\lambda > 0$ (refer [10]).

Lemma 2.11. [12] *Let B be a non-empty closed convex subset of real Hilbert space M. Let $g : B \to (-\infty, \infty)$ is proper convex and lower semi continuous. For every $r > 0$, $\lambda > 0$, the following holds:*

$$J_r^g x = J_r^g x(\frac{\lambda}{r}x + (1 - \frac{\lambda}{r}))J_r^g x.$$

Lemma 2.12. [6] *Let B be a non-empty closed convex subset of real Hilbert space M. Let $g : B \to (-\infty, \infty)$ be a proper convex and lower semi continuous. For every $x, y \in M$, $\lambda > 0$, the following holds:*

$$\frac{1}{\lambda}||J_\lambda^g x - y||^2 - \frac{1}{\lambda}||x - y||^2 + \frac{1}{\lambda}||x - J_\lambda^g x||^2 + g(J_\lambda^g x) \leq g(y).$$

Lemma 2.13. [9] *Let $\{t_k\}$ be a sequence of real numbers which does not decrease at infinity in the sense that there exists a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that*

$t_k \leq t_{k_i+1}$ for all $i \geq 0$. For sufficiently large $k \in \mathbb{N}$, let $\{\tau_k\}$ be the sequence of integers defined as follows:

$$\tau(k) = \max\{k \leq m : t_k \leq t_k + 1\}.$$

Then $\tau(k) \to \infty$ as $k \to \infty$ and $\max\{t_{\tau(k)}, t_k\} \leq t_{\tau(k)+1}$.

Proposition 2.14. [17] *Let (X, d) be a complete CAT(0) space and let $g : X \to (-\infty, \infty)$ be a proper convex and lower semi-continuous function. Let U be a nonexpansive mapping on X such that $F(U) \cap F(J_\lambda) \neq \emptyset$ for all λ. Then for, any $\lambda > 0$, $F(U \circ J_\lambda) = F(U) \cap F(J_\lambda)$.*

Lemma 2.15. [22] *Let (X, d) be a complete CAT(0) space and let B be non-empty closed convex subset of X, and $U : B \to B$ be a nonexpansive mapping. Let $u \in B$ be fixed. For each $t \in (0, 1)$, the mapping $v_t : B \to B$ defined by $v_t x = tu \oplus (1 - t)Ux$, for $x \in B$ has a unique fixed point $y_t \in B$, i.e., $y_t = v_t y_t = tu \oplus (1 - t)Uy_t$.*

3. Main Results

Theorem 3.1. *Let B be a non-empty closed convex subset of a real Hilbert space M and $g : B \to (-\infty, \infty)$ is a proper convex and lower semi-continuous. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $U : B \to B$ is nonexpansive mappings such that $\Omega = F(U) \cap argmin_{u \in B} g(u) \neq \emptyset$. Let $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $0 < \eta < \frac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta(k - \frac{L^2 \eta}{2})$ and $I - U$ is demiclosed at the origin. Let $\{x_k\}$ be a sequence defined by (1.6), where $\{\alpha_k\}, \{\beta_k\}$ are sequences in $[0, 1]$ such that $\lim_{k \to \infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k \to \infty} \inf \beta_k (1 - \beta_k) > 0$ and $\{\lambda_k\}$ is a sequence such that $0 < \lambda \leq \lambda_k$ for all $k \geq 1$ and for some λ. Then the sequence $\{x_k\}$ generated by (1.6) converges strongly to $x' \in \Omega$ which is also a unique solution of the variational inequality problem (VIP):*

$$(\eta A x' - \gamma f(x'), x' - p) \leq 0. \tag{3.1}$$

Proof. Suppose that $\alpha_k \in (0, \min\{1, \frac{1}{\tau}\})$. We prove the the theorem in the following manner:

(i) The sequence $\{x_k\}$ generated by (1.6) is bounded.
(ii) $x' \in \Omega$ is the unique solution of the variational inequality problem (3.1).
(iii) The sequence $\{x_k\}$ converges strongly to $x' \in \Omega$.

(i) Since $\Omega = F(U) \cap argmin_{u \in B} g(u) \neq \emptyset$, so suppose that $p \in \Omega$ and, hence $g(p) \leq g(u)$ for all $u \in B$. This implies that

$$g(p) + \frac{1}{2\lambda_k} ||p - p||^2 \leq g(u) + \frac{1}{2\lambda_k} ||u - p||^2.$$

Hence $J^g_{\lambda_k}p = p$ for all $k \geq 1$. Using nonexpansivenes of $J^g_{\lambda_k}$, we have

$$\begin{aligned}
||z_k - p|| &= ||J^g_{\lambda_k}x_k - J^g_{\lambda_k}p|| \\
&\leq ||x_k - p||, \\
||w_k - p|| &= ||Uz_k - p|| \\
&\leq ||x_k - p||, \\
||v_k - p|| &= ||(1-\beta_k)x_k + \beta_k w_k - p|| \\
&\leq (1-\beta_k)||x_k - p|| + \beta_k||x_k - p|| \\
&\leq ||x_k - p||.
\end{aligned}$$

By using Lemma 2.10, we have

$$\begin{aligned}
||x_{k+1} - p|| &= ||P_K(\alpha_k\gamma f(x_k) + (I - \eta\alpha_k A))v_k - p|| \\
&\leq ||(\alpha_k\gamma f(x_k) + (I - \eta\alpha_k A)v_k - p|| \\
&= ||\alpha_k\gamma f(x_k) + (I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p + (I - \eta\alpha_k A)p - p|| \\
&\leq ||\alpha_k\gamma f(x_k) + (I - \eta\alpha_k A)p - p|| + ||(I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p|| \\
&= ||\alpha_k\gamma f(x_k) - \eta\alpha_k Ap|| + ||(I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p|| \\
&\leq (1 - \alpha_k\tau)||v_k - p|| + \alpha_k\gamma||f(x_k) - f(p)|| + ||\alpha_k\gamma f(p) - \alpha_k\eta Ap|| \\
&\leq (1 - \alpha_k\tau)||x_k - p|| + \alpha_k\gamma b||x_k - p|| + \alpha_k||\gamma f(p) - \eta Ap|| \\
&\leq (1 - \alpha_k(\tau - \gamma b))||x_k - p|| + \alpha_k||\gamma f(p) - \eta Ap|| \\
&\leq \max\{||x_k - p||, \frac{||\gamma f(p) - \eta Ap||}{\tau - \gamma b}.
\end{aligned}$$

Applying induction, we get

$$||x_{k+1} - p|| \leq \max\{||x_0 - p||, \frac{||\gamma f(p) - \eta Ap||}{\tau - \gamma b}.$$

for all $k \geq 1$. This implies that $\{x_k\}$ is bounded.

(ii) By using Lemma 2.8, we have

$$\begin{aligned}
||v_k - p||^2 &= ||(1-\beta_k)x_k + \beta_k w_k - p||^2 \\
&= (1-\beta_k)||x_k - p||^2 + \beta_k||w_k - p||^2 - \beta_k(1-\beta_k)||x_k - w_k||^2 \\
&= (1-\beta_k)||x_k - p||^2 + \beta_k||Tz_k - p||^2 - \beta_k(1-\beta_k)||x_k - w_k||^2 \\
&\leq (1-\beta_k)||x_k - p||^2 + \beta_k||z_k - p||^2 - \beta_k(1-\beta_k)||x_k - w_k||^2 \\
&\leq ||x_k - p||^2 - \beta_k(1-\beta_k)||x_k - w_k||^2.
\end{aligned}$$

$$\begin{aligned}
||x_{k+1} - p||^2 &\leq ||(\alpha_k\gamma f(x_k) + (I - \eta\alpha_k A)v_k - p||^2 \\
&\leq ||\alpha_k\gamma f(x_k) + (I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p + (I - \eta\alpha_k A)p - p||^2 \\
&\leq (1 - \alpha_k\tau)^2||v_k - p||^2 + \alpha_k^2||\gamma f(x_k) - \eta Ap||^2 \\
&+ 2\alpha_k(1 - \tau\alpha_k)||\gamma f(x_k) - \eta Ap||.||v_k - p|| \\
&\leq \alpha_k^2||\gamma f(x_k) - \eta Ap||^2 + (1 - \alpha_k\tau)^2[||x_k - p||^2 - \alpha_k(1-\alpha_k)(1 - \tau\alpha_k)^2||x_k - w_k||^2] \\
&+ 2\alpha_k(1 - \tau\alpha_k)||\gamma f(x_k) - \eta Ap||.||v_k - p||.
\end{aligned}$$

Since $\{x_k\}$ is bounded sequence , so suppose that

$$(1-\tau\alpha_k)^2||x_k-w_k||^2 \le ||x_k-p||^2 - ||x_{k+1}-p||^2 + 2\alpha_k D, \tag{3.2}$$

where $D > 0$.
Now consider VIP: find $x' \in \Omega$ such that

$$(\eta Ax' - \gamma f(x'), x' - p) \le 0, \ \forall p \in \Omega. \tag{3.3}$$

We will show the uniqueness of the solution of (3.3). Let $x', x^* \in \Omega$ are the solutions of (3.3), then

$$(\eta Ax' - \gamma f(x'), x' - x^*) \le 0$$

and

$$(\eta Ax^* - \gamma f(x^*), x^* - x') \le 0$$

subtracting above, we get

$$\begin{aligned}&(\eta Ax^* - \eta Ax' - \gamma f(x^*) + \gamma f(x'), x^* - x' - x' + x^*) \le 0\\ &\Rightarrow (\eta Ax^* - \eta Ax' - \gamma f(x^*) + \gamma f(x'), x^* - x') \le 0.\end{aligned}$$

Note that

$$\begin{aligned}\frac{L^2}{2}\eta > 0 &\Longleftrightarrow k - \frac{L^2}{2}\eta < k\\ &\Longleftrightarrow \eta(k - \frac{L^2}{2}\eta) < \eta k\\ &\Longleftrightarrow \tau < \eta k\\ &\Rightarrow 0 < b\gamma < \tau < \eta k.\end{aligned}$$

Also note that

$$(\eta Ax^* - \eta Ax' - \gamma f(x^*) + \gamma f(x'), x^* - x') \ge (\eta k - b\gamma)||x^* - x'||^2,$$

this implies that $x^* = x'$.

(iii) We claim that $P_\Omega(I + (t\gamma f - \eta tA))$ is contraction mapping. For this, let $x, y \in M$, and $t \in \mathbb{R}$ such that $t \in (0, \min\{1, \frac{1}{\tau}\})$. Then

$$\begin{aligned}||P_\Omega(I + (t\gamma f - \eta tA))x - P_\Omega(I + (t\gamma f - \eta tA))y|| &\le ||(I + (t\gamma f - \eta tA))x\\ &\quad - (I + (t\gamma f - \eta tA))y||\\ &= ||t\gamma f(x) - t\gamma f(y)||\\ &\quad + ||(I - t\eta A))x - (I - t\eta A)y||\\ &\le t\gamma b||x-y|| + (1-\tau t)||x-y||\\ &= (1 - t(\tau - \gamma b)||x-y||\\ &\le \delta||x-y||,\end{aligned}$$

where $\delta = (1 - t(\tau - \gamma b)$. It concludes that $P_\Omega(I + (t\gamma f - \eta tA))$ is a contraction mapping. So by Banach contraction principle, $P_\Omega(I + (t\gamma f - \eta tA))$ has a unique fixed point say z in M. That is $z = P_\Omega(I + (t\gamma f - \eta tA))z$. In the view of inequality (2.4), it is equivalent to the following VIP:

$$(\eta Az - \gamma f(z), z - p) \le 0 \, \forall \, z \in \Omega.$$

By uniqueness of the solution of (3.1), we have $z = x^*$.
To show that the sequence $\{x_k\}$ converges strongly to $x' \in \Omega$. Consider the following cases:
Case I: when the sequence $\{||x_k - p||\}$ is monotonically decreasing. Since $\{||x_k - p||\}$ is bounded, hence $\{||x_k - p||\}$ is convergent and

$$\lim_{k\to\infty} ||x_k - p||^2 - ||x_{k+1} - p||^2 = 0.$$

So from (3.2), we have

$$\lim_{k\to\infty} (1 - \tau\alpha_k)^2 ||x_k - w_k||^2 = 0,$$

which implies that $||x_k - w_k||^2 = 0$ as $\lim_{k\to\infty} \alpha_k = 0$. Also by Lemma 2.12 and since $g(p) \leq g(z_k)$, we have for $J^g_\lambda x = x_k$ and $y = z_k$

$$\begin{aligned}
\frac{1}{\lambda}||x_k - z_k||^2 - \frac{1}{\lambda}||x - z_k||^2 + \frac{1}{\lambda}||x - x_k||^2 + g(x_k) \leq g(z_k)& \\
\Rightarrow ||x_k - z_k||^2 \leq ||x - z_k||^2 + ||x - x_k||^2& \\
+ \lambda g(x_k) \leq \lambda g(z_k)& \\
\Rightarrow ||x_k - z_k||^2 \leq ||x - z_k||^2 + ||x - x_k||^2.&
\end{aligned}$$

By using Lemma 2.8, we have

$$\begin{aligned}
||x_{k+1} - p||^2 &\leq ||(\alpha_k \gamma f(x_k) + (I - \eta\alpha_k A))v_k - p||^2 \\
&\leq ||\alpha_k \gamma f(x_k) + (I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p + (I - \eta\alpha_k A)p - p||^2 \\
&= ||((I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p + (I - \eta\alpha_k A)p) \\
&+ \alpha_k \gamma f(x_k) + (p - \eta\alpha_k Ap) - p||^2 \\
&\leq ||((I - \eta\alpha_k A)v_k - (I - \eta\alpha_k A)p||^2 + 2(\alpha_k \gamma f(x_k) + p - \\
&\eta\alpha_k Ap, (I - \eta\alpha_k A)v_k - ((I - \eta\alpha_k A)p \\
&+ \alpha_k \gamma f(x_k) + p - \eta\alpha_k Ap - p) \\
&\leq (1 - \tau\alpha_k)^2 ||v_k - p||^2 + 2(\alpha_k \gamma f(x_k) - \eta\alpha_k Ap, x_{k+1} - p) \\
&\leq (1 - \tau\alpha_k)^2 ||x_k - p||^2 + 2\alpha_k(\gamma f(x_k) - \eta Ap, x_{k+1} - p) \\
&\leq (1 - \tau\alpha_k)^2 ||x_k - p||^2 + 2\alpha_k(\gamma f(x_k) - \gamma f(p) + \gamma f(p) - \eta Ap, x_{k+1} - p) \\
&\leq (1 - \tau\alpha_k)^2 ||x_k - p||^2 + 2\alpha_k \gamma ||f(x_k) - f(p)||.||x_{k+1} - p|| \\
&+ 2\alpha_k ||\gamma f(p) - \eta Ap||.||x_{k+1} - p|| \\
&\leq (1 - \tau\alpha_k)^2 ||x_k - p||^2 + 2\alpha_k \gamma b ||x_k - p||.||x_{k+1} - p|| \\
&+ 2\alpha_k ||\gamma f(p) - \eta Ap||.||x_{k+1} - p||.
\end{aligned}$$

Hence

$$\begin{aligned}
(1 - \tau\alpha_k)^2 ||x_k - z_k||^2 \leq\ & ||x_k - p||^2 + ||x_{k+1} - p||^2 + \tau^2 \alpha_k ||x_k - p||^2 + \\
& 2\alpha_k \gamma b ||x_k - p||.||x_{k+1} - p|| + 2\alpha_k ||\gamma f(p) - \eta Ap||.||x_{k+1} - p|| \\
& \Rightarrow 0 \ as\ k \to \infty.
\end{aligned}$$

Next, we prove that

$$\limsup_{k\to\infty}(\eta Ax^* - \gamma f(x^*), x^* - x_k) \leq 0.$$

Since M is Hilbert space and $\{x_k\}$ is bounded sequence, there exists a subsequence say $\{x_{k_n}\}$ of $\{x_k\}$ such that it converges weakly to $m \in B$ and

$$\limsup_{k\to\infty}(\eta Ax^* - \gamma f(x^*), x^* - x_k) = \limsup_{k\to\infty}(\eta Ax^* - \gamma f(x^*), x^* - x_{k_n}).$$

Since $I-U$ is demiclosed, we have $m \in F(U)$. Now by using Lemma 2.11, we have

$$\begin{aligned}
||x_k - J_\lambda^g x_k|| &\leq ||x_k - z_k|| + ||z_k - J_\lambda^g x_k|| \\
&\leq ||x_k - z_k|| + ||J_{\lambda_k}^g x_k - J_\lambda^g x_k|| \\
&\leq ||x_k - z_k|| + ||J_\lambda^g(\frac{\lambda}{\lambda_k}x_k + (1-\frac{\lambda}{\lambda_k})J_{\lambda_k^g}x_k) - J_\lambda^g x_k|| \\
&\leq ||x_k - z_k|| + (1-\frac{\lambda}{\lambda_k})||z_k - x_k|| \\
&\leq (2-\frac{\lambda}{\lambda_k})||x_k - z_k|| \\
&\Rightarrow ||x_k - J_\lambda^g x_k|| \to 0 \ as\ k \to \infty.
\end{aligned}$$

Since $I-U$ is demiclosed and J_λ^g is nonexpansive, we have $m \in F(J_\lambda^g) = argmin_{u\in B} g(u)$. Therefore $m \in \Omega$. Hence

$$\begin{aligned}
\limsup_{k\to\infty}(\eta Ax^* - \gamma f(x^*), x^* - x_k) &= \limsup_{k\to\infty}(\eta Ax^* - \gamma f(x^*), x^* - x_{k_n}) \\
&= (\eta Ax^* - \gamma f(x^*), x^* - m) \leq 0.
\end{aligned}$$

Now

$$\begin{aligned}
||x_{k+1} - x'||^2 &\leq ||(\alpha_k \gamma f(x_k) + (I - \eta\alpha_k A)v_k - x'||^2 \\
&\leq ||\alpha_k \gamma f(x_k) + (I - \eta\alpha_k A)v_k \\
&\quad - (I - \eta\alpha_k A)p + (I - \eta\alpha_k A)p - x'||^2 \\
&\leq (1 - (\tau - b\gamma)\alpha_k)^2 ||x_k - x'||^2 + 2\alpha_k(\eta Ax - \gamma f(x), x' - x_{k+1}) \\
&\to 0 \ as\ k \to \infty.
\end{aligned}$$

Hence $x_k \to x'$.

Case II: when $\{||x_k - p||\}$ is not monotonically decreasing sequence. Let $C_k = ||x_k - p||$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $k \geq k_0$ by $\tau(k) = \max\{k \in \mathbb{N} : k \leq m, C_k \leq C_{k+1}\}$. Clearly $\{\tau\}$ is non-decreasing sequence such that $\tau(k) \to \infty$ as $k \to \infty$ and $C_{\tau(k)} \leq C_{\tau(k)+1}$ for all $k \geq k_0$. From (3.2), we have

$$\begin{aligned}
(1 - \tau\alpha_{\tau(k)})^2 ||x_k - w_k||^2 &\leq 2\alpha_{\tau(k)} D \to 0 \ as\ k \to \infty \\
&\Rightarrow ||x_k - w_k|| \to 0 \ as\ k \to \infty.
\end{aligned}$$

By doing similar argument as in Case I, we have

$$\limsup_{k\to\infty}(\eta Ax^* - \gamma f(x^*), x^* - x_{\tau(k)}) \leq 0.$$

Thus for all $k \geq k_0$, we have

$$\begin{aligned}0 \leq ||x_{\tau(k)+1} - x^{'}||^2 - ||x_{\tau(k)} - x^{'}||^2 &\leq \alpha_{\tau(k)}(-(\tau - b)\gamma))||x_{\tau(k)} - x^{'}||^2 \\ &\quad + 2\alpha_{\tau(k)}(\eta Ax^{'} - \gamma f(x^{'}), x^{'} - x_{\tau(k)+1}) \\ ||x_{\tau(k)} - x^{'}||^2 &\leq \frac{2}{(\tau - b\gamma)}(\eta Ax^{'} - \gamma f(x^{'}), x^{,} - x_{\tau(k)+1}) \\ &\to 0 \ as \ k \to \infty.\end{aligned}$$

Therefore $\lim_{k\to\infty} C_{\tau(k)} = \lim_{k\to\infty} C_{\tau(k)+1} = 0$. Then by Lemma 2.13, we have $0 \leq c_k \leq \max\{C_{\tau(k)}, C_{\tau(k)+1}\} = C_{\tau(k)+1}$. Hence $\lim_{k\to\infty} C_k = 0$, that is $\{x_k\}$ converges strongly to $x^{'}$. □

Corollary 3.2. *Let B be a non-empty closed convex subset of a real Hilbert space M and $g : B \to (-\infty, \infty)$ is proper convex and lower-semi continuous. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $U_k : B \to B$ is finite family of nonexpansive mappings such that $\Omega = F(U) \cap argmin_{u\in B} g(u) \neq \emptyset$ for all $p \in \Omega$, where $F(U) = \cap_{k=1}^{n} F(U_k)$. Let $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $0 < \eta < \frac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta(k - \frac{L^2\eta}{2})$ and $I - U_k$ are demiclosed at the origin for $k = 1, 2, ..., n$. Let $\{x_k\}$ be a sequence defined by (1.6), where $\{\alpha_k\}, \{\beta_k\}$ are sequence in $[0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \inf \beta_k(1 - \beta_k) > 0$ and $\{\lambda_k\}$ is a sequence such that $0 < \lambda \leq \lambda_k$ for all $k \geq 1$ and for some λ. Then the sequence $\{x_k\}$ generated by (1.6) converges strongly to $x^{'} \in \Omega$, which is a unique solution of the variational inequality problem:*

$$(\eta Ax^{'} - \gamma f(x^{'}), x^{'} - p) \leq 0.$$

Theorem 3.3. *Let B be a non-empty closed convex subset of a real Hilbert space M and $g : B \to (-\infty, \infty)$ is proper convex and lower semi-continuous. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $U : B \to B$ is nonexpansive mappings such that $\Omega = F(U) \cap argmin_{u\in B} g(u) \neq \emptyset$ for all $p \in \Omega$. Let $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $0 < \eta < \frac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta(k - \frac{L^2\eta}{2})$ and $I - U$ is demiclosed at the origin. Let $\{x_k\}$ be a sequence defined by (1.6), where $\{\alpha_k\}, \{\beta_k\}$ are sequence in $[0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \inf \beta_k(1 - \beta_k) > 0$ and $\{\lambda_k\}$ is a sequence such that $0 < \lambda \leq \lambda_k$ for all $k \geq 1$ and for some λ. Then the sequence $\{x_k\}$ generated by (1.6) converges strongly to $x^{'} \in \Omega$, which is together a minimizer of g in B and a fixed point of U.*

Proof. Since $\{x_k\}$ is bounded, therefore $\{v_k\}$, $\{J_{\lambda_k} x_k\}$ all are bounded. Also $||x_{k+1} - x||^2 \to 0$ as $k \to \infty$,, we have $\lim_{k\to\infty} ||x_{k+1} - x_k|| = 0$. Also $\lim_{k\to\infty} ||x_k -$

$w_k|| = 0$. Now by nonexpansiveness of U, we have

$$\begin{aligned}||U \circ J_\lambda^g x_k - x_k|| &\leq ||U \circ J_\lambda^g x_k - w_k|| + ||w_k - x_k||\\ &\leq ||U \circ J_\lambda^g x_k - U z_k|| + ||w_k - x_k||\\ &\leq ||J_\lambda^g x_k - z_k|| + ||w_k - x_k||\\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Now for $t \in (0,1)$, let $v_t : B \to B$ such that $v_t = (1-t)x + (U \circ J_\lambda^g)x$. Then from Lemma 2.15, we conclude that v_t has a fixed point say $z \in F(U \circ J_\lambda^g)$, i.e., $z = F(U \circ J_\lambda^g)z$. Also by Proposition 2.14, we have $F(U \circ J_\lambda^g) = F(U) \cap F(J_\lambda^g)$, z is together a minimizer of g in B and a fixed point of U. □

Corollary 3.4. *Let B be a non-empty closed convex subset of a real Hilbert space M and $g : B \to (-\infty, \infty)$ is proper convex and lower semi-continuous. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $U_k : B \to B$ is family of nonexpansive mappings such that $\Omega = F(U) \cap argmin_{u \in B} g(u) \neq \emptyset$ for all $p \in \Omega$, where $F(U) = \cap_{k=1}^n F(U_k)$. Let $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $0 < \eta < \frac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta(k - \frac{L^2 \eta}{2})$ and $I - U_k$ is demiclosed at the origin for $k = 1, 2, ..., n$. Let $\{x_k\}$ be a sequence defined by (1.6), where $\{\alpha_k\}, \{\beta_k\}$ are sequence in $[0,1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \inf \beta_k(1-\beta_k) > 0$ and $\{\lambda_k\}$ is a sequence such that $0 < \lambda \leq \lambda_k$ for all $k \geq 1$ and for some λ. Then the sequence $\{x_k\}$ generated by (1.6) converges strongly to $x' \in \Omega$, which is together a minimizer of g in B and a fixed point of U.*

Corollary 3.5. *Let B be a non-empty closed convex subset of a real Hilbert space M and $g : B \to (-\infty, \infty)$ is proper convex and lower semi-continuous. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $U : B \to B$ is $k-$ strictly pseudo-contractive mappings such that $\Omega = F(U) \cap argmin_{u \in B} g(u) \neq \emptyset$ for all $p \in \Omega$. Let $A : B \to M$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $0 < \eta < \frac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta(k - \frac{L^2 \eta}{2})$ and $I - U$ is demiclosed at the origin. Let $\{x_k\}$ be a sequence defined by (1.6), where $\{\alpha_k\}, \{\beta_k\}$ are sequence in $[0,1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \inf \beta_k(1-\beta_k) > 0$ and $\{\lambda_k\}$ is a sequence such that $0 < \lambda \leq \lambda_k$ for all $k \geq 1$ and for some λ. Then the sequence $\{x_k\}$ generated by (1.6) converges strongly to $x' \in \Omega$, which is together a minimizer of g in B and a fixed point of U.*

Proof. Since every nonexpansive mapping is strictly pseudo-contractive mapping, therefore result follows from the Theorem 3.1 □

Corollary 3.6. *Let B be a non-empty closed convex subset of a real Hilbert space M and $g : B \to (-\infty, \infty)$ is proper convex and lower semi-continuous. Let $f : B \to M$ be a $b-$ Lipschitzian mapping and $U_k : B \to B$ is family of $k-$ strictly pseudo-contractive mappings such that $\Omega = F(U) \cap argmin_{u \in B} g(u) \neq \emptyset$ for all $p \in \Omega$, where $F(U) = \cap_{k=1}^n F(U_k)$. Let $A : B \to B$ be a $k-$ strongly monotone and $L-$ Lipschitzian mapping. Let $0 < \eta < \frac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta(k - \frac{L^2 \eta}{2})$ and $I - U_k$ is demiclosed at the origin for $k = 1, 2, ..., n$. Let $\{x_k\}$ be a sequence defined by (1.6), where $\{\alpha_k\}, \{\beta_k\}$ are sequence in $[0,1]$ such that*

$\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \inf \beta_k(1-\beta_k) > 0$ *and* $\{\lambda_k\}$ *is a sequence such that* $0 < \lambda \leq \lambda_k$ *for all* $k \geq 1$ *and for some* λ. *Then the sequence* $\{x_k\}$ *generated by* (1.6) *converges strongly to* $x' \in \Omega$, *which is together a minimizer of* g *in* B *and a fixed point of* U.

4. Application in Split feasibility problem

In 1994, Censor and Elfving [4] introduced split feasibility problem (SFP). Due to its applications (SFP) like in image reconstruction, signal processing, and intensity-modulated radiation therapy (refer [25, 26]), several authors gained attention in this direction.

The split feasibility problem is to find a point x with the property that

$$x \in B \text{ and } Dx \in Q, \tag{4.1}$$

where B and Q are non-empty closed convex subset of Hilbert spaces M_1, M_2 and $D : M_1 \to M_2$ is bounded linear operator.

We say that a point x^* is the solution of (4.1) if and only if $x^* \in B$ and $Ax^* - P_Q Dx^* = 0$. The proximality function g is defined by

$$g(x) = \frac{1}{2}||Dx - P_Q Dx||^2. \tag{4.2}$$

Now consider convex minimization problem

$$\min_{x\in B} g(x) = \min_{x\in B} \frac{1}{2}||Dx - P_Q Dx||^2.$$

Then x^* solves SFP (4.1) if and only if x^* solves convex minimization problem (4.2).

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Department of Mathematics, Government Dudhadhari Bajrang Girls' Post-graduate Auotnomous College, Raipur-492001, Chhattisgarh, India

Email address: dewangan.kiran@gmail.com