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ISOGEOMETRIC ANALYSIS APPROXIMATION OF PARABOLIC PROBLEM WITH RIGHT-HAND SIDE IN L^1

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Abstract. In the present paper, we study approximation by a new space-time Isogeometric Analysis (IgA) method for linear parabolic evolution equation namely

$$\frac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) = f \text{ in } Q = \Omega \times (0,T).$$

We assume that the physical mesh is quasi-uniform. In addition, the right-hand side f of the problem belongs to $L^1(Q)$. We prove that the unique solution of the discrete problem, obtained by the Isogeometric Analysis method, converges in $L^1(0,T;W_0^{1,q}(\Omega))$, for every q with $1 \leq q < \dfrac{d}{d-1}$, to the unique renormalized solution of the problem.

1. Introduction

The Isogeometric analysis method was introduced in [11] as a new discretization approach for Partial Differential Equations (PDEs). The core idea of IgA is to use the same smooth and hight-order superior finite dimensional B-spline or Non-Uniform Rational B-Splines (NURBS) spaces for parametrizing the computational domain and for approximating the solution of the PDE model under consideration. IgA approximates have suitably been applied to the solution of a large number of linear and nonlinear problems (see [5]). Approximation's properties of B-splines and their applications for discretizing PDEs have been given in [17].

In this paper, we present a new stable space-time Isogeometric Analysis method. We focus the single-patch to space-time IgA schemes. We consider the linear parabolic initial-boundary value problems : find $u : \overline{Q} \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) & = f \quad \text{in } Q = \Omega \times (0,T), \\ \qquad u & = 0 \quad \text{on } \mathit{\Sigma} = \partial\Omega \times (0,T)\ , \\ \qquad u & = u_0 \quad \text{on } \mathit{\Sigma}_0 = \partial\Omega \times \{0\}. \end{cases} \tag{1.1}$$

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This is a typical parabolic model problem posed in the space- time cylinder $\overline{Q} = \overline{\Omega} \times [0,T] = Q \cup \Sigma \cup \Sigma_0 \cup \Sigma_T$, where $\Sigma_T := \Omega \times \{T\}$ and $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ denotes the spatial computational domain with the boudary $\partial\Omega$.
The main contribution of this paper is to consider that the right-hand side f and the data u_0, respectively, belongs to $L^1(Q)$ and $L^1(\Sigma_0)$.
When the source function $f \in L^2(Q)$ and the initial conditions $u_0 \in L^2(\Omega)$, the existence and the uniqueness of a space-time variational formulation were investigated in [12] and [13]. In our case where f and u_0 belong to $L^1(\Omega \times (0,T))$ and $L^1(\Sigma_0)$, respectively, we cannot use the notion of variational formulation. To solve this problem, we consider in the present paper the framework of renormalized solutions. This notion was introduced by DiPerna and Lions in [10] and Dall'Aglio (see [9]). The interested reader can also see [4].
In our recent work [1], we have studied the approximation of elliptic problem, namely $-\mathrm{div}(A\nabla u) = f$ with the right-hand side f belongs to $L^1(\Omega)$. We have established that the unique solution of the discrete problem, obtained by the Isogeometric Analysis method, converges to the unique renormalized solution of the problem.
Here, we consider the parabolic case. This problem modelling a heat tranfer in the space-time cylinder Q.
The standard space-time isogeometric analysis scheme of the problem (1.1), given by

$$\left\{\begin{array}{l} \text{Find } u_h \in V_{0h} \text{ such that } \forall v_h \in V_{0h}, \\ \displaystyle\int_Q \partial_t u_h v_h dxdt + \int_Q \theta h \partial_t u_h \cdot \partial_t v_h dxdt + \int_Q A\nabla_x u_h \cdot \nabla_x v_h dxdt \\ \displaystyle\quad +\theta h \int_Q A\nabla_x u_h \cdot \nabla_x \partial_t v_h dxdt = \int_Q f\Big(v_h + \theta h \partial_t v_h\Big) dxdt, \end{array}\right. \tag{1.2}$$

where

$$V_{0h} = \Big\{ v_h \in V_h : v_h = 0 \text{ on } \Sigma \cup \Sigma_0 \Big\},$$

has a unique solution (see Proposition 2.4 below). Note that V_h is the NURBS space defined in (2.19).
As the solution of (1.1) does not belong to $L^2(0,T;H_0^1(\Omega))$ for usually a general right-hand side $f \in L^1(Q)$, we do not have the insurance that the solution of (1.2) converges in $L^2(0,T;H_0^1(\Omega))$ towards to solution u of (1.1).
Hence the need to use the concept of renormalized solution. This definition of the renormalized solution allow one to shown that in this new sense problem (1.1) is well posed in the terminology of Hadamard, namely that the solution of (1.1) exists, is unique, and depends continuously on the right-hand side f.
The existence and the uniqueness of renormalized solution of (1.1) are proved in [3].
In the present paper, we prove that the unique solution u_h of (1.2) converges to the unique renormalized solution u of (1.1) in the following sense

$$u_h \longrightarrow u \text{ strongly in } L^1(0,T;W_0^{1,q}(\Omega)) \tag{1.3}$$

for every q with $1 \leq q < \dfrac{d}{d-1}$.
To achieve this section, we assume that the physical mesh family is quasi-uniform. The rest of the paper is organized as follows: In Section 2, we make preliminaries that we used in this paper. we also recall the framework of Isogeometric analysis method. In Section 3 , we prove our main result and give an error estimates in Section 4. For this, we introduce the Marcinkiewicz space still calling weak-Lebesgue space. This is a set of all measurable functions $f \in Q$ which satistying

$$\|f\|_{L^{r,\infty}(Q)} := \sup_{\lambda>0} \lambda \Big|\{z \in Q \ : \ |f(z)| > \lambda\}\Big|^{1/r} < +\infty. \tag{1.4}$$

Finally, we give numerical implementation in the last Section.

2. **Preliminaries**

2.1. **Assumptions.** We consider the following parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A\nabla u) & = f \quad \text{in } Q = \Omega \times (0,T), \\ \qquad u & = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T)\ , \\ \qquad u & = u_0 \quad \text{on } \Sigma_0 = \partial\Omega \times \{0\}, \end{cases} \tag{2.1}$$

under the following assumptions:
$A(x,t)$ is a symmetric coercive matrix with coefficients lying in $L^\infty(Q)$ such that

$$A \in L^\infty(Q)^{d\times d} \text{ and } a_{ij}(x,t) = a_{ji}(x,t) \text{ a.e. in } Q,\ \forall i,j, \tag{2.2}$$

satisfying the ellipticity assumption

$$\exists \alpha > 0, \text{ a.e. in } Q,\ \forall \xi \in \mathbb{R}^d,\ A(x,t)\xi\cdot\xi \geq \alpha|\xi|^2 \tag{2.3}$$

and

$$\exists \beta > 0, \text{ a.e. in } Q,\ \forall \xi \in \mathbb{R}^d,\ A(x,t)\xi\cdot\partial_t\xi \geq \beta|\partial_t\xi|^2. \tag{2.4}$$

The data of this problem are such that

$$f \in L^1(Q), \qquad u_0 \in L^1(\Sigma_0). \tag{2.5}$$

∇ is the gradient operator and the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is fixed, bounded and Lipschitz. We assume that the space-time cylinder $\overline{Q} = \overline{\Omega} \times [0,T] = Q \cup \Sigma \cup \Sigma_0 \cup \Sigma_T$,
where

$$Q := \Omega \times (0,T), \quad \Sigma := \partial\Omega \times (0,T), \quad \Sigma_0 := \Omega \times \{0\}, \quad \Sigma_T := \Omega \times \{T\}.$$

Troughout this paper and for any positive real number k, we denote by T_k the truncation function at height k as the real-valued Lipschitz function such that

$$T_k(r) = \min\Big(k,\ \max(r,-k)\Big).$$

Now, we are able to define the renormalized solution of our problem.

Definition 2.1. A real valued function u defined on $\Omega\times(0,T)$ is a renormalized solution of (2.1) if

$$u\in L^\infty(0,T;L^1(\Omega)); \tag{2.6}$$

$$T_k(u)\in L^2(0,T;H_0^1(\Omega)); \tag{2.7}$$

$$\lim_{h\longrightarrow+\infty}\int_0^T\int_{\{h\le|u|\le k+h\}}|\nabla u|^2dxdt=0,\ \text{for any } k>0; \tag{2.8}$$

and

$$\begin{aligned}-\int_0^T\langle S(u),\xi_t\rangle dt+\int_0^T\int_\Omega S'(u)A\nabla u\,\xi dxdt\\ +\int_0^T\int_\Omega S''(u)A\nabla u\nabla\xi dxdt=\int_0^T\int_\Omega fS'(u)\xi dxdt,\end{aligned} \tag{2.9}$$

for every function $\xi\in C_c^\infty(\Omega\times[0,T))$ and $S\in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support and $S'(u)\xi\in L^2(0,T;H_0^1(\Omega))$.

Conditions (2.6) and (2.7) permit to define ∇u almost everywhere in Q: for any $k>0$, we have $\nabla T_k(u)=\chi_{\{|u|<k\}}\nabla u$ a.e. in Q where $\chi_{\{|u|<k\}}$ designates the characteristic function of the set $\{(x,t);|u(x,t)<k|\}$.
Equation (2.9) can be formally obtained through pointwise multiplication of (2.1) by $S'(u)$.

Theorem 2.2. *(see [3]) Assume that f satisfies condition (2.5). Then, there exists a unique renormalized solution of (1.1). Moreover this unique solution belongs to $L^q(0,T;W^{1,q}(\Omega))$ for every q with $1\le q<\dfrac{d}{d-1}$. It depends continuously on the right-hand side f in the following sense: if f^ε is a sequence which satisfies*

$$f^\varepsilon\longrightarrow f \text{ strongly in } L^1(Q) \tag{2.10}$$

when ε tends to zero, then the sequence u^ε of the renormalized solution of (1.1) for the right-hand sides f^ε satisfies for every $k>0$ and for every q with $1\le q<\dfrac{d}{d-1}$,

$$T_k(u^\varepsilon)\longrightarrow T_k(u) \text{ strongly in } L^2(0,T;H_0^1(\Omega)),$$
$$u^\varepsilon\longrightarrow u \text{ strongly in } L^q(0,T;W_0^{1,q}(\Omega)),$$

when ε tends to zero, where u is the renormalized solution of (1.1) for the right-hand side f. Finally, if f_1 and f_2 belong to $L^1(Q)$, and if u_1 and u_2 are the renormalized solutions of (1.1) for the right-hand sides f_1 and f_2 respectively, then for every $k>0$, the function $T_k(u_1-u_2)$ belongs to $L^2(0,T;H_0^1(\Omega))$ and for every q with $1\le q<\dfrac{d}{d-1}$, one has

$$\begin{aligned}\|T_k(u_1-u_2)\|^2_{L^2(0,T;H_0^1(\Omega))}\le k\|f_1-f_2\|_{L^1(Q)},\\ \|u_1-u_2\|^2_{L^q(0,T;W_0^{1,q}(\Omega))}\le C_1\|f_1-f_2\|_{L^1(Q)},\end{aligned} \tag{2.11}$$

where the constant C_1 only depends on $d,|\Omega|$ and q.

In the following subsection, we give some essential result of the method of Isogeometric Analysis.

2.2. **Stable Space – Time IgA Discretizations.** We introduce new space-time IgA schemes for our model problem. We briefly present the notions of B-splines and NURBS basis functions. In the sequel, we consider the construction of the IgA spaces by using the NURBS space. For more details on B-splines and NURBS-based IgA, we refer to [8, 15, 16]. We start with the B-spline spaces. We denote

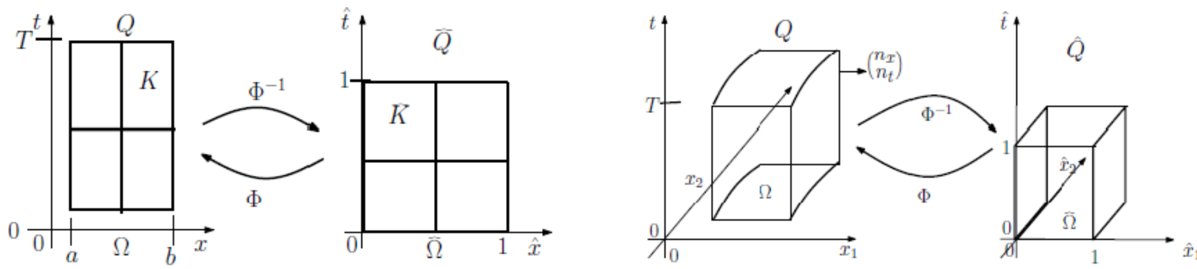

FIGURE 1. Space-time IgA paraphernalia for $\mathbb{Q} \subset \mathbb{R}^{d+1}$ with $d = 1$ (left) and $d = 2$ (right).

by the integer $p \geq 1$ the polynomial degree and n the number of basis functions. Then a knot vector is nothing but a set of non-decreasing sequence of real numbers in the parameter domain and can be written as $\varXi = \left\{\xi_1, ..., \xi_{n+p+1}\right\}$. We also assume, by convention, that $\xi_1 = 0$ and $\xi_{n+p+1} = 1$. We also note that the knot value may be repeated indicating its multiplicity m. We always consider open knot vectors, i.e., knot vectors where the first and last knot values appear $p+1$ times, or, in other words, the multiplicity m of the first and last knots is $p+1$. For the one-dimensional parametric domain $(0, 1)$, there is an underlying mesh $\widehat{\mathcal{K}}_h$ consisting of elements $\widehat{K}$ created by the distinct knots. We denote the global mesh size by $\hat{h} := \max\{\hat{h}_K : \widehat{K} \in \widehat{\mathcal{K}}_h\}$, where $\hat{h}_K := diam(\widehat{K}) = length(\widehat{K})$. For the time being fixed, we assume that the ratio of the sizes of neighboring elements are uniformly bounded from above and below. In this case, we speak about locally quasi-uniform meshes. The univariate B-spline basis functions $\widehat{B}_{i,p} : (0, 1) \longrightarrow \mathbb{R}$ are recursively defined by means of the Cox-de Boor recursion formula:

$$\widehat{B}_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & otherwise. \end{cases} \tag{2.12}$$

For $p = 1, 2, 3, ...$, they are defined by

$$\widehat{B}_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}\widehat{B}_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}}\widehat{B}_{i+1,p-1}(\xi). \tag{2.13}$$

By definition, we suppose that a division by zero is equal to be zero. We observe that a basis function of degree p is $(p-m)$ times continuously differentiable across a knot with the multiplicity m. If all internal knots have the multiplicity 1, then B-splines of degree p are globaly C^{p-1}-continuous.
Now, we can define the multivariate B-spline basis functions on the space-time parameter domain $\widehat{Q} := (0,1)^{d+1} \subset \mathbb{R}^{d+1}$ with $d = 1, 2$ or 3. It is a product of the corresponding univariate B-spline basis functions. To do this, we first define the knot vectors $\mathit{\Xi}_\alpha = \left\{\xi_1^\alpha, ..., \xi_{n_\alpha+p_\alpha+1}^\alpha\right\}$ for every space-time direction $\alpha = 1, ..., d+1$.
Additionaly, we consider the multi-indices $i := (i_1, ..., i_{d+1})$ and $p := (p_1, ..., p_{d+1})$, the set $\overline{\mathcal{I}} = \{i = (i_1, ..., i_{d+1}) :\ i_\alpha = 1, 2, ..., n_\alpha; \alpha = 1, 2, ..., d+1\}$. The product of the univariate B-spline basis functions allows to obtain immediatly multivariate B-spline basis functions

$$\widehat{B}_{i,p}(\xi) := \prod_{\alpha=1}^{d+1} \widehat{B}_{i_\alpha,p_\alpha}(\xi_\alpha), \tag{2.14}$$

where we take $\xi = (\xi_1, ..., \xi_{d+1}) \in \widehat{Q}$. The univariate and multivariate NURBS basis functions are defined in the parametric domains by means of the corresponding B-spline basis functions $\{\widehat{B}_{i,p}\}_{i\in\overline{\mathcal{I}}}$. For given $p = (p_1, ..., p_{d+1})$ and for all $i \in \overline{\mathcal{I}}$, we define the NURBS basis functions $\widehat{R}_{i,p}$ as follows:

$$\widehat{R}_{i,p} : (0,1) \longrightarrow \mathbb{R},\ \widehat{R}_{i,p}(\xi) := \frac{\widehat{B}_{i,p}(\xi)w_i}{W(\xi)} \tag{2.15}$$

with the weighting function

$$W : (0,1) \longrightarrow \mathbb{R},\ W(\xi) := \sum_{i\in\overline{\mathcal{I}}} w_i\widehat{B}_{i,p}(\xi), \tag{2.16}$$

where the weights w_i are positive real numbers.
In this paper, we describe the space-time cylinder $Q \subset \mathbb{R}^{d+1}$ by considering the parametric domain $\widehat{Q} = (0,1)^{d+1}$ and the geometrical mapping

$$\Phi : \widehat{Q} \longrightarrow Q = \Phi(\widehat{Q}) \subset \mathbb{R}^{d+1},\ \Phi(\xi) = \sum_{i\in\overline{\mathcal{I}}} \widehat{R}_{i,p}(\xi)\mathbf{P}_i \tag{2.17}$$

where $\widehat{R}_{i,p},\ i \in \overline{\mathcal{I}}$, are the NURBS basis functions and $\{\mathbb{P}_i\}_{i\in\overline{\mathcal{I}}} \subset \mathbb{R}^{d+1}$ are the control points.

By means of the geometrical mapping (2.17), we define the physical mesh $\mathcal{K}_h$, associated to the computational domain Q, whose vertices and elements are the images of the vertices and elements of the corresponding underlying tensor-product mesh $\widehat{\mathcal{K}}_h$ in the parametric domain $\widehat{Q}$, i.e., $\mathcal{K}_h := \{K = \Phi(\widehat{K}) : \widehat{K} \in \widehat{\mathcal{K}}_h\}$. We denote the global mesh size of the mesh in the physical domain by $h := \max\{h_K : K \in \mathcal{K}_h\}$, with $h_K := \|\nabla\Phi\|_{L^\infty(K)}\widehat{h}_{\widehat{K}}$ and $\widehat{h}_{\widehat{K}} = diam(\widehat{K})$.
Moreover, we assume that there exists a positive constant C_u, independent of h, such that

$$h_K \leq h \leq C_u h_K \qquad \text{for all } K \in \mathcal{K}_h. \tag{2.18}$$

The previous relationship (2.18) implies that the physical mesh is quasi uniform. Finally, we define the NURBS space

$$V_h = \operatorname{span}\{\varphi_{h,i} = \widehat{R}_{i,p} \circ \Phi^{-1}\}_{i \in \overline{\mathcal{I}}} \tag{2.19}$$

on Q by a push-forward of the NURBS basis functions, where we assume that the geometrical mapping Φ is invertible a.e. in Q, with smooth inverse on each element K of the physical mesh $K \in \mathcal{K}_h$, see [2] and [18] for more details.
Since we discretize the whole space-time cylinder Q by means of NURBS, we will introduce the space-time IgA space V_{0h} by using the NURBS V_h:

$$V_{0h} := V_h \cap H^{1,1}_{0,\underline{0}}(Q) = \Big\{ v_h \in V_h : v_h|_{\sum \cup \sum_0} = 0 \Big\} = \operatorname{span}\{\varphi_{h,i}\}_{i \in \mathcal{I}}, \tag{2.20}$$

where

$$H^{1,1}_{0,\underline{0}}(Q) = \Big\{ u \in L^2(Q) : \nabla_x u \in \Big[L^2(Q)\Big]^d,\ u = 0 \text{ on } \Sigma \text{ and } u = 0 \text{ on } \Sigma_T \Big\}.$$

For simplicity, we assume below the same polynomial degree for all coordinate directions, i.e. $p_\alpha = p$ for all $\alpha = 1, 2, ..., d+1$.
Now, we recall some approximation properties of B-splines (resp. NURBS) space. In ([2], Lemma 3.2 and Lemma 3.3), a projector on the B-spline space V_{0h} is introduced. The projector, here denoted by Π_h, with the present notation is written as

$$\Pi_h v := \sum_{i_1=1,\dots,i_d=1}^{m_1,\dots,m_d} (\lambda_{i_1\dots i_d} v),\ \forall v \in L^2((0,1)^d),$$

where the $\lambda_{i_1\dots i_d}$ are *dual* basis functionals, that is,

$$\begin{cases} \lambda_{j_1\dots j_d} B_{i_1\dots i_d} &= 1 \quad \text{if } j_\alpha = i_\alpha,\ \forall 1 \leq \alpha \leq d, \\ \lambda_{j_1\dots j_d} B_{i_1\dots i_d} &= 0 \quad \text{otherwise.} \end{cases}$$

From [[17], chapter 12], the functionals $\lambda_{i_1\dots i_d}$ can be represented by functions with local support. This induces local stability properties on Π_h. We summarize the previous properties in the following result, proved in [[17], Theorem 2.6].

Lemma 2.3. *We have*

$$\Pi_h v_h = v_h,\ \forall v_h \in V_{0h} \text{ (spline preserving)},$$

$$\|\Pi_h v\|_{L^2(K)} \leq C\|v\|_{L^2(K)}\ \forall v \in L^2((0,1)^d), \forall K \ \in\ \mathcal{K}_h \text{ (stability)}.$$

2.3. **Tools.** Here, we show various results which will be used in particular in the proof of our main theorem.
Motivated by the bilinear form of the left-hand side of (1.2), we introduce the mesh-dependent norm

$$\|v_h\|_h = \Big[\|\nabla_x v_h\|^2_{L_2(Q)} + \theta h\|\partial_t v_h\|^2_{L_2(Q)} + \frac{1}{2}\|v_h\|^2_{L_2(\Sigma_T)}\Big]^{1/2}. \tag{2.21}$$

To obtain our discrete scheme for defining an approximate solution $u_h \in V_{0h}$, we multiply our parabolic problem (2.1) by a time-upwind test function of the form $v_h + \theta h\partial_t v_h$ with an arbitrary $v_h \in V_{0h}$. The constant θ is positive and depends on the degree p, but its also independent of h (see [14] for more details).
In the next result, we show that the standard Isogeomertic analysis scheme of (2.1) exists and is unique.

Proposition 2.4. *Under the assumptions* (2.2)-(2.5)*, the standard Isogeometric Analysis scheme of* (2.1) *given by*

$$\left\{\begin{aligned} &\text{Find } u_h \in V_{0h} \text{ such that } \forall v_h \in V_{0h}\\ &\int_Q \partial_t u_h v_h dxdt + \theta h\int_Q \partial_t u_h\cdot\partial_t v_h dxdt + \int_Q A\nabla_x u_h\cdot\nabla_x v_h dxdt\\ &\quad +\theta h\int_Q A\nabla_x u_h\cdot\nabla_x\partial_t v_h dxdt = \int_Q f\Big(v_h + \theta h\partial_t v_h\Big)dxdt, \end{aligned}\right. \tag{2.22}$$

has a unique solution.

Proof. Since $v_h \in V_{0h} \subset L^\infty(Q)$ and f belongs to $L^1(Q)$, the right-hand side of (2.22) makes sense. Let $a_h(.,.) : V_{0h}\times V_{0h} \longrightarrow \mathbb{R}$ the discrete continuous and bilinear form defined by

$$\begin{aligned} a_h(u_h, v_h) \;\;=\;\; &\int_Q \partial_t u_h v_h dxdt + \theta h\int_Q \partial_t u_h\cdot\partial_t v_h dxdt\\ &+\int_Q A\nabla_x u_h\cdot\nabla_x v_h dxdt + \theta h\int_Q A\nabla_x u_h\cdot\nabla_x\partial_t v_h dxdt \end{aligned}$$

and the continuous and linear form

$$l_h(v_h) = \int_Q \big(v_h + \theta h\partial_t v_h\big) f dxdt.$$

It's remind to establish that the bilinear form is $V_{0h}-$coercive wrt the norm $\|.\|_h$ for all $v_h \in V_{0h}$.

We have, by using asumptions (2.3) and (2.4)

$$\begin{aligned}
a_h(v_h, v_h) &\geq \int_Q \Big(\frac{1}{2}\partial_t v_h^2 + \theta h(\partial_t v_h)^2 \ + \alpha|\nabla_x v_h|^2 + \beta\theta h|\partial_t \nabla_x v_h|^2\Big) dxdt \\
&\geq \frac{1}{2}\int_{\partial Q} v_h^2 \eta_t ds + \theta h\|\partial_t v_h\|^2_{L^2(Q)} \ + \alpha\|\nabla_x v_h\|^2_{L^2(Q)} \\
&\geq \frac{1}{2}\|v_h\|^2_{L_2(\Sigma_T)} + \theta h\|\partial_t v_h\|^2_{L_2(Q)} + \alpha\|\nabla_x v_h\|^2_{L_2(Q)} \\
&\geq C\left(\frac{1}{2}\|v_h\|^2_{L_2(\Sigma_T)} + \theta h\|\partial_t v_h\|^2_{L_2(Q)} + \|\nabla_x v_h\|^2_{L_2(Q)}\right) \\
&\geq C\|v_h\|^2_h,
\end{aligned}$$

where C depends only on α. So, the bilinear form $a_h(.,.)$ is V_{0h}-elliptic.
The V_{0h}-ellipticity of the bilinear form $a_h(.,.)$ implies uniqueness of the solution u_h of the IgA scheme (2.22). Therefore, as V_{0h} is a finite dimension space, uniqueness yields existence. Thus, there always exists a unique IgA solution of the space-time IgA scheme (2.22). □

We need the following result.

Theorem 2.5. *Assume that $v_h \in V_{0h}$ satisfies*

$$\forall k > 0, \ \int_Q \Big|\nabla_x \Pi_h(T_k(v_h))\Big|^2 dxdt \leq kM, \tag{2.23}$$

for some $M > 0$. Then, for every q with $1 \leq q < \dfrac{d}{d-1}$,

$$\|v_h\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C_2\, M, \tag{2.24}$$

where the constant C_2 only depends on d, $|\Omega|$, and q.

Proof. For the proof, we adapt the technique used in [7] (see Theorem 2.1).
We consider Friedrich's inequality that we use in the form

$$\|v\|_{L^{2^*}(Q)} \leq C_\Omega\|\nabla_x v\|_{L^2(Q)},$$

which holds for all $v \in H^1(Q)$ which vanishing trace on Σ. The proof of Friedrichs' inequality is given in [6]. Moreover, we have firstly $2^* = \dfrac{2d}{d-2}$ if $d \geq 3$ (in this case, C_Ω depends only on d). Secondly, 2^* is any real number with $1 \leq 2^* < +\infty$ if $d = 2$ (and then the constant C_Ω depends on $|\Omega|$).
From this estimate and (2.23), we obtain that

$$\int_Q \Big|\Pi_h(T_k(v_h))\Big|^{2^*} dxdt \leq C_\Omega^{2^*}\Big(\int_Q \Big|\nabla_x \Pi_h(T_k(v_h))\Big|^2 dxdt\Big)^{\frac{2^*}{2}} \leq C_\Omega^{2^*}(kM)^{\frac{2^*}{2}}. \tag{2.25}$$

For $k > 0$, we define the set E_k by

$$E_k = \bigcup \Big\{K \in \mathcal{K}_h : \exists (y,t) = z \in K \text{ with } |v_h(z)| \geq k\Big\}.$$

According to ([7], Lemma 2.3), for every $K \in \mathcal{K}_h$, with $K \subset E_k$, there exists a micro element $E \subset K$, with $|E| = C(d)|K|$ and

$$\forall x \in E, \ \Big|\Pi_h(T_k(v_h)(x))\Big| \geq \frac{k}{2}.$$

Consequently, for $K \subset E_k$

$$\int_K \Big|\Pi_h(T_k(v_h))\Big|^{2*} dxdt \geq \int_E \Big|\Pi_h(T_k(v_h))\Big|^{2*} dxdt \geq \Big(\frac{k}{2}\Big)^{2*}|E| \geq \Big(\frac{k}{2}\Big)^{2*} C(d)|K|,$$

so

$$\begin{aligned} |E_k| = \sum_{K\subset E_k} |K| \ &\leq \ \sum_{K\subset E_k} \frac{1}{C(d)\Big(\frac{k}{2}\Big)^{2*}} \int_K \Big|\Pi_h(T_k(v_h))\Big|^{2*} dxdt \\ &\leq \ \frac{1}{C(d)\Big(\frac{k}{2}\Big)^{2*}} \int_Q \Big|\Pi_h(T_k(v_h))\Big|^{2*} dxdt. \end{aligned}$$

Considering (2.25), we obtain

$$|E_k| \leq \frac{1}{C(d)\Big(\frac{k}{2}\Big)^{2*}} C_\Omega^{2*}(kM)^{\frac{2*}{2}} = \frac{C_\Omega^{2*}(kM)^{\frac{2*}{2}}}{C(d)} \frac{M^{\frac{2*}{2}}}{k^{\frac{2*}{2}}}. \tag{2.26}$$

The inclusion $\{(x,t) \in Q : |v_h(x,t)| \geq k\} \subset E_k$ and inequality (2.26) imply that

$$k^{\frac{2^*}{2}}\Big|\{(x,t) \in Q : |v_h(x,t)| \geq k\}\Big| \leq k^{\frac{2^*}{2}}|E_k| \leq \frac{2C_\Omega^{2^*}}{C(d)} M^{\frac{2^*}{2}}.$$

As $d \geq 3$ implies $\dfrac{2^*}{2} = \dfrac{d}{d-2}$, this last inequality gives,

$$\|v_h\|_{L^{\frac{d}{d-2},\infty}(Q)} \leq C(d)M. \tag{2.27}$$

Here, $C(d)$ denotes a constant which only depends on d.
We take $z = (x,t) \in Q$. For every $\lambda > 0$ and for every $k > 0$, one has

$$\begin{aligned} \Big\{z \in Q : |\nabla_x v_h(z)| \geq \lambda\Big\} = \Big\{z \in Q : |\nabla_x v_h(z)| \geq \lambda \text{ and } z \in E_k\Big\} \cup \\ \cup\Big\{z \in Q : |\nabla_x v_h(z)| \geq \lambda \text{ and } z \in E_k^c\Big\}. \end{aligned}$$

Hence, we obtain

$$\Big|\Big\{z \in Q : |\nabla_x v_h(z)| \geq \lambda\Big\}\Big| \leq \Big|E_k\Big| + \Big|\Big\{z \in Q : |\nabla_x v_h(z)| \geq \lambda \text{ and } z \in E_k^c\Big\}\Big|. \tag{2.28}$$

For each $K \in E_k^c$, we have $|v_h(z)| \leq k$ and consequently, $\Pi_h(T_k(v_h))(z) = v_h(z)$ and $\nabla_x \Pi_h(T_k(v_h))(z) = \nabla_x v_h(z)$.

So we can write that

$$\begin{aligned}
&\text{meas}\Big\{z\in Q:|\nabla_x v_h(z)|\geq\lambda \text{ and } z\in E_k^c\Big\}\\
&\qquad=\text{meas}\Big\{z\in Q:|\nabla_x\Pi_h(T_k(v_h))(z)|\geq\lambda \text{ and } z\in E_k^c\Big\}\\
&\qquad\leq\text{meas}\Big\{z\in Q:|\nabla_x\Pi_h(T_k(v_h))(z)|\geq\lambda\Big\}\\
&\qquad\leq\frac{1}{\lambda^2}\int_Q\Big|\nabla_x\Pi_h(T_k(v_h))(z)\Big|^2dxdt.
\end{aligned}$$

Considering (2.28), (2.27) and the hypothesis (2.23), we obtain that for $\lambda>0$ and every $k>0$,

$$\Big|\Big\{z\in Q:|\nabla_x v_h(z)|\geq\lambda\Big\}\Big|\leq\frac{(2C_\Omega)^{2^*}}{C(d)}\Big(\frac{M}{k}\Big)^{\frac{2^*}{2}}+\frac{kM}{\lambda^2}.$$

Taking $k=\lambda^{\frac{4}{2^*+2}}+M^{\frac{2^*-2}{2^*+2}}$, we have

$$\lambda^{\frac{22^*}{2^*+2}}\Big|\{z\in Q:|\nabla_x v_h(z)|\geq\lambda\Big\}\Big|\leq\Big(\frac{(2C_\Omega)^{2^*}}{C(d)}+1\Big)M^{\frac{22^*}{2^*+2}}.$$

This means that, when $d\geq 3$

$$\|\nabla_x v_h\|_{L^{\frac{d}{d-1},\infty}(Q)^d}\leq C(d)M, \tag{2.29}$$

which implies (2.24).
When $d=2$, we use the estimate (2.27) and the embedding inequality

$$\forall p,\ 1\leq p<r,\ \|u\|_{L^p(Q)}\leq C(p,r,|\Omega|)\|u\|_{L^{r,\infty}(Q)} \tag{2.30}$$

to obtain (2.24). □

Proposition 2.6. *Under the assumptions (2.2)-(2.5) and (2.18), the solution u_h of (2.22) satisfies for every $h>0$ and every $k>0$*

$$\|v_h\|_{L^1(0,T;W_0^{1,q}(\Omega))}\leq C_3(d,|\Omega|,q)\|f\|_{L^1(Q)}. \tag{2.31}$$

where the constant $C_3(d,|\Omega|,q)$ only depends on d, $|\Omega|$ and q.

Proof. Since $T_k(u_h)$ is continuous, the function $\Pi_h(T_k(u_h))$ belongs to V_{0h}.
We can write that

$$a_h\Big(\Pi_h(T_k(u_h)),\Pi_h(T_k(u_h))\Big)=\int_Q f\Big[\Pi_h(T_k(u_h))+\theta h\partial_t\Big(\Pi_h(T_k(u_h))\Big)\Big]dxdt. \tag{2.32}$$

On one part, we consider the left hand side of (2.32).
Taking $\Lambda_{h,k}=\Pi_h(T_k(u_h))$, we obtain after using asumptions (2.3) and (2.4)

$$a_h\Big(\Lambda_{h,k},\Lambda_{h,k}\Big)\ \geq\ \int_Q\Big(\frac{1}{2}\partial_t\Lambda_{h,k}^2+\theta h(\partial_t\Lambda_{h,k})^2\ +\alpha|\nabla_x\Lambda_{h,k}|^2+\beta\theta h|\partial_t\nabla_x\Lambda_{h,k}|^2\Big)dxdt$$

for all $\Lambda_{h,k} \in V_{0h}$. Hence, we get

$$\alpha \int_Q |\nabla_x \Lambda_{h,k}|^2 dxdt \leq a_h\Big(\Lambda_{h,k}, \Lambda_{h,k}\Big).$$

On the other hand, we have

$$\int_Q f\Big[\Pi_h(T_k(u_h)) + \theta h \partial_t\Big(\Pi_h(T_k(u_h))\Big)\Big] dxdt \leq (k_1 + \theta h k_2)\|f\|_{L^1(Q)}.$$

Furthermore, we obtain

$$\alpha \int_Q \Big|\nabla_x \Pi_h(T_k(u_h))\Big|^2 dxdt \leq C(h,k,\theta)\|f\|_{L^1(Q)} \tag{2.33}$$

and by Theorem (2.5), we obtain, for each $T > 0$,

$$\int_0^T \|u_h\|_{W_0^{1,q}(\Omega)} dt \leq C(h,k,\theta) \int_0^T \|f\|_{L^1(\Omega)} dt.$$

Finally, we get

$$\|u_h\|_{L^1(0,T;W_0^{1,q}(\Omega))} \leq C(h,k,\theta)\|f\|_{L^1(Q)},$$

which completes the proof. □

3. Convergence Analysis

Theorem 2.5 implies that u_h is bounded in $L^1(0,T;W_0^{1,q}(\Omega))$ for every q with $1 \leq q < \dfrac{d}{d-1}$. We prove, in this part the strong convergence of u_h in this space.

3.1. Main result.

The following theorem is our main result.

Theorem 3.1. *Under the assumptions* (2.2)-(2.5) *and* (2.18)*, the solution u_h of* (2.22) *satisfies for every q with $1 \leq q < \dfrac{d}{d-1}$,*

$$u_h \longrightarrow u \text{ strongly in } \; L^1(0,T;W_0^{1,q}(\Omega)), \tag{3.1}$$

when h tends to zero, where u is the unique renormalized solution of (1.1).

Proof. We adapt here the techniques used in [7] for the proof of Theorem 3.2. We take a sequence f^ε of functions such that

$$f^\varepsilon \in L^\infty(Q), \qquad f^\varepsilon \longrightarrow f \text{ strongly in } L^1(Q).$$

As we have the inclusion $L^\infty \subset L^2(Q)$, this sequence is obtained by taking for example $f^\varepsilon = T_{\frac{1}{\varepsilon}}(f)$.

We consider the unique solution u_h^ε of (2.22) with the right-hand side f^ε. Subsequently, we get

$$\begin{cases} u_h - u_h^\varepsilon \ \in V_{0h}, \\ \forall v_h \ \in V_{0h}, \displaystyle\int_Q \partial_t(u_h - u_h^\varepsilon) v_h dxdt + \theta h \int_Q \partial_t(u_h - u_h^\varepsilon)\cdot\partial_t v_h dxdt \\ + \displaystyle\int_Q A\nabla_x(u_h - u_h^\varepsilon)\cdot\nabla_x v_h dxdt + \theta h \int_Q A\nabla_x(u_h - u_h^\varepsilon)\cdot\nabla_x\partial_t v_h dxdt \\ \qquad\qquad = \displaystyle\int_Q (f - f^\varepsilon) v_h dxdt + \theta h \int_Q (f - f^\varepsilon)\partial_t v_h dxdt. \end{cases} \tag{3.2}$$

Applying estimate (2.33) to this problem, we obtain for every $k > 0$, every $h > 0$ and every $\varepsilon > 0$

$$\alpha \int_0^T \int_\Omega \left|\nabla_x T_k(u_h - u_h^\varepsilon)\right|^2 dxdt \le k\|f - f^\varepsilon\|_{L^1(Q)},$$

which implies by Theorem 2.5 (see [7], Theorem 2.1) that for every q with $1 \le q < \dfrac{d}{d-1}$, every $h > 0$ and every $\varepsilon > 0$

$$\int_0^T \|u_h - u_h^\varepsilon\|_{W_0^{1,q}(\Omega)} dt \le C_3(d, |\Omega|, q)\|f - f^\varepsilon\|_{L^1(Q)}. \tag{3.3}$$

On the other hand, since $f^\varepsilon \in L^2(Q)$ and considering that the family of physical mesh $\mathcal{K}_h$ satisfies (2.18), we have that for every fixed ε

$$u_h^\varepsilon \longrightarrow u^\varepsilon \text{ strongly in } L^2(0,T; H_0^1(\Omega)), \tag{3.4}$$

where h tends to zero, where u^ε is the unique solution of

$$\begin{cases} u^\varepsilon \ \in \ L^2(0,T; H_0^1(\Omega)), \\ \dfrac{\partial u^\varepsilon}{\partial_t} - \operatorname{div}(A\nabla u^\varepsilon) = f^\varepsilon \text{ in } \mathcal{D}'(Q). \end{cases} \tag{3.5}$$

At the end, the function u^ε is respectively the unique weak solution of (3.5) as well the unique renormalized solution in the sense of Definition 2.1 of the problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial_t} - \operatorname{div}(A\nabla u^\varepsilon) & = & f^\varepsilon \text{ in } Q, \\ u^\varepsilon & = & 0 \text{ on } \Sigma, \\ u^\varepsilon & = & u_0^\varepsilon \text{ on } \Sigma_0. \end{cases} \tag{3.6}$$

By application of the estimate (2.11), we have

$$\|u^\varepsilon - u\|_{L^1(0,T;W_0^{1,q}(\Omega))} \le C_4(d, |\Omega|, q)\|f^\varepsilon - f\|_{L^1(Q)}, \tag{3.7}$$

for every q with $1 \le q < \dfrac{d}{d-1}$, where u is the unique renormalized solution of (1.1). Now, we can use the triangular inequality in the following sense, and for all $T>0$

$$\int_0^T \|u_h-u\|_{W_0^{1,q}(\Omega)}dt \leq \int_0^T \|u_h-u_h^\varepsilon\|_{W_0^{1,q}(\Omega)}dt+\int_0^T \|u_h^\varepsilon-u^\varepsilon\|_{W_0^{1,q}(\Omega)}dt + \int_0^T \|u^\varepsilon-u\|_{W_0^{1,q}(\Omega)}dt.$$

Using (3.3), (3.4) and (3.7), we obtain, for every $\varepsilon>0$ and every q with $1\leq q<\dfrac{d}{d-1}$,

$$\limsup_{h\longrightarrow 0}\|u_h-u\|_{L^1(0,T;W_0^{1,q}(\Omega))}\leq \Big(C_3(d,|\Omega|,q)+C_4(d,|\Omega|,q)\Big)\|f^\varepsilon-f\|_{L^1(Q)}.$$

To complete this proof, we take in previous estimate the limit when ε tends to zero and we obtain (3.1). □

3.2. **Discretization Error estimate.** When $f\in L^1(Q)$, the precedent Theorem 3.1 establishes the convergence of the IgA schemes method, but it does not provide any error estimates. Here, we assume that f belongs to the Marcinkiewicz space $L^{r,\infty}(Q)$ (with $r>1$) and that Ω is sufficiently smooth.

Theorem 3.2. *Under the assumptions of Theorem 3.1, if we further assume that either $d=2$ or $d=3$, that $f\in L^{r,\infty}(Q)$ for some r with $1<r<2$, that Ω is a convex polyhedron, then we have the error estimate*

$$\|u_h-u\|_{L^1(0,T;W_0^{1,q}(\Omega))}\leq C(d,q,r,\Omega,)h^{2(1-\frac{1}{r})}\|f\|_{L^{r,\infty}(Q)}. \tag{3.8}$$

Proof. We consider that f belongs to the Marcinkiewicz space $L^{r,\infty}(Q)$ for some r with $1<r<2$. We introduce the approximate problem of (2.22). Let $\varepsilon>0$, we consider

$$f^\varepsilon=T_{\frac{1}{\varepsilon}}(f).$$

We obtain $\{f^\varepsilon\}\subset L^\infty(Q)\subset L^2(Q)$ such that

$$f^\varepsilon\longrightarrow f \text{ strongly in } L^1(Q),$$

as ε tends to zero. We denote by u_h^ε the solution of this problem with right-hand side f^ε. Defining also u^ε at the solution of (3.6), we have for every q with $1\leq q<\dfrac{d}{d-1}$,

$$\int_0^T \|u_h-u\|_{W_0^{1,q}(\Omega)}dt \leq \int_0^T \|u_h-u_h^\varepsilon\|_{W_0^{1,q}(\Omega))}dt+\int_0^T \|u_h^\varepsilon-u^\varepsilon\|_{W_0^{1,q}(\Omega)}dt + \int_0^T \|u^\varepsilon-u\|_{W_0^{1,q}(\Omega)}dt. \tag{3.9}$$

Our approximate problem with right-hand side $f^\varepsilon\in L^2(Q)$ has an unique solution u_h^ε and satisfies

$$\|u_h^\varepsilon-u^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}\leq Ch\|f^\varepsilon\|_{L^2(Q)},$$

where the constant $C > 0$ is independent of h and f^ε. Therefore, from the continuous imbedding of $H_0^1(\Omega)$ in $W_0^{1,q}(\Omega)$, we obtain a new constant C (which depends on q, Ω) such that

$$\|u_h^\varepsilon - u^\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq Ch\|f^\varepsilon\|_{L^2(Q)}.$$

Using then (3.3) and (3.7), we deduce that for a new constant C, which is not depend of ε, h and f (but depends on d, q, Ω),

$$\|u_h - u\|_{L^1(0,T;W_0^{1,q}(\Omega))} \leq C\Big(\|f - f^\varepsilon\|_{L^1(Q)} + h\|f^\varepsilon\|_{L^2(Q)}\Big). \tag{3.10}$$

But now, we're focusing on the right-hand side. We give an estimate of this term. For this, we use the definition of the norm in the Marcinkiewicz space $L^{r,\infty}(Q)$ and the embedding inequality (see [7], Remark 2.2).

$$\forall q,\ 1 \leq q < r,\ \|\psi\|_{L^q(Q)} \leq C\Big(q, r, |\Omega|\Big)\|\psi\|_{L^{r,\infty}(Q)}.$$

We obtain, on the one hand

$$\|f - f^\varepsilon\|_{L^1(Q)} \leq \frac{1}{r-1}\varepsilon^{r-1}\|f\|^r_{L^{r,\infty}(Q)} \tag{3.11}$$

and furthermore

$$\|f^\varepsilon\|_{L^2(Q)} \leq \sqrt{\frac{2}{2-r}}\frac{1}{\varepsilon^{1-\frac{r}{2}}}\|f\|^{\frac{r}{2}}_{L^{r,\infty}(Q)}. \tag{3.12}$$

Therefore, the estimate (3.10) allows to obtain

$$\|u_h - u\|_{L^1(0,T;W_0^{1,q}(\Omega))} \leq C\Big(\frac{1}{r-1}\varepsilon^{r-1}\|f\|^r_{L^{r,\infty}(Q)} + \sqrt{\frac{2}{2-r}}\frac{h}{\varepsilon^{1-\frac{r}{2}}}\|f\|^{\frac{r}{2}}_{L^{r,\infty}(Q)}\Big).$$

To end this proof, we take in this inequality $\varepsilon = \dfrac{h^{\frac{2}{r}}}{\|f\|^{\frac{r}{2}}_{L^{r,\infty}(Q)}}$, for every q with $1 \leq q < \dfrac{d}{d-1}$ and for every $h > 0$, and we obtain finally

$$\|u_h - u\|_{L^1(0,T;W_0^{1,q}(\Omega))} \leq C\Big(d, q, r, \Omega,\Big)h^{2(1-\frac{1}{r})}\|f\|^r_{L^{r,\infty}(Q)}.$$

□

4. Numerical results

We present here some numerical experiments to validate the theoretical estimates presented in Section 4. The examples considered are motivated by [[14], Section 5], where the authors used Isogeometric Analysis (IgA).

We consider the space-time domain $Q := \Omega(t) \times [0,1] \subset \mathbb{R}^2$ where $\Omega(t) = \{x \in \mathbb{R} : a(t) < x < b(t)\}$ with $t = (0,1),\ a(t) = -t/2$ and $b(t) = 1 + t/2$. We will consider a deforming space-time domain with polygonal surface area Σ as illustrated in Fig.2. By using unstructured mesh with $p = 1$ and $p = 2$ polynomial degree continuous finite elements, we compute the error in the norm $\|.\|_{r,\infty}$.

The renormalized solution of the problem (2.1) is given by $u(x,t) = \sin(\pi x)\sin(\pi t)$. We choose the positive parameter $\theta = 0.1$ and we set

$$f(x) = \frac{1}{|x|^{2/r}} \ \in \ L^{r,\infty}(\Omega(t)).$$

We have estimated the convergence order in $L^1(0,T;W_0^{1,1}(\Omega))$ norm and we take $r = 1.5$. The solution contours u_h can be seen in Fig.3.

We study the case $p = 1$ and $p = 2$. We perform a convergence analysis where the mesh is successively refined in the *Degree of Freedom*. In any case, we observe that the error diminishes as the discretization order is increased using the k-refinement previously defined.

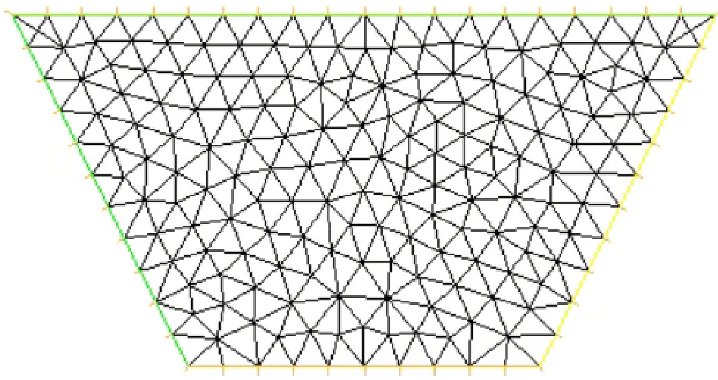

FIGURE 2. Moving spatial domains with underlying unstrunctured mesh

5. Concluding remarks

In this article, we have presented a new stable space-time finite element scheme for parabolic evolution problems with L^1 data on arbitrary space-time domains. We have presented *a priori* error estimates in the Marcinkiewicz space and numerical examples in the space-time domain $Q \subset \mathbb{R}^2$. For this, we use continuous finite element approximations with polynomial degrees $p = 1$ and $p = 2$.

The new results present the possibility for space-time adaptativity in moving domains, also the possibility to use preconditionning, fast solvers like multi-grid and domain decomposition solvers.

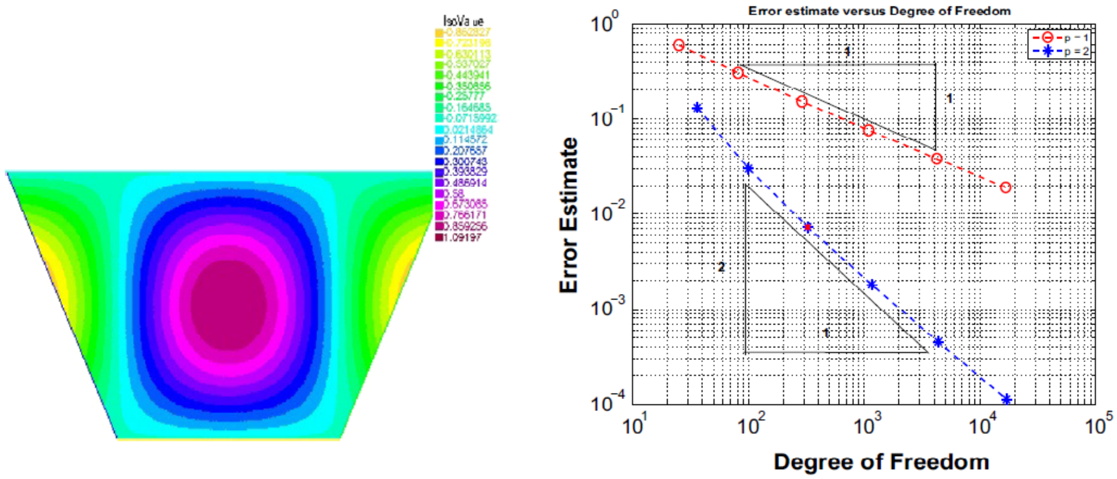

FIGURE 3. Solution contours in the space-time cylinder Q (left) and the plot of the estimates for degrees $p = 1$ and $p = 2$

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