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COUPLED FIXED POINT RESULTS FOR CONTRACTIVE TYPE CONDITIONS IN PARTIAL b-METRIC SPACES

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Abstract. The aim of this paper is to establish some coupled fixed point results for contractive type conditions in the framework of partial b-metric spaces. Further, we give some consequences of the established results. We also prove well-posedness of fixed point problem of the mapping. The results of findings in this paper extend and generalize several previously published findings from the existing literature (see, for example, [1], [19] and many others).

1. Introduction and preliminaries

There are a number of generalizations of metric spaces and Banach contraction principle. In this sequel, Bakhtin [2] and Czerwik [6] introduced b-metric spaces as a generalization of metric spaces (see, also [7]). They proved the contraction mapping principle in b-metric spaces that generalized the well-known Banach contraction principle in such spaces. Since then, several papers have dealt with fixed point theory for single-valued and multi valued operators in b-metric spaces.

In 1994, *Matthews* (see, [13, 14]) introduced the notion of partial metric spaces as a part of the study of denotational semantics of data flow networks. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any pint of space may not be zero. Further, *Matthews* showed that the Banach contraction principle [3] is valid in partial metric space. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see, e.g., [9], [17] and some others). In 1999, Heckmann [9] generalized it by omitting the small self-distance axiom. The partial metric defined by Heckmann is called weak partial metric.

On the other hand, *Bhashkar and Lakshmikantham* [4] introduced the concept of coupled fixed point of a mapping $\psi\colon \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. As, an application, they studied the existence and uniqueness of the solution for a

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periodic boundary value problem associated with first order differential equation. Later, several other authors such as *Ciric and Lakshmikantham* [5], *Sabetghadam et al.* [19] and *Olaleru et al.* [17] proved some coupled fixed point theorems in metric spaces (see, also [8], [10], [11], [12], [15], [16], [20], [21]). *Aydi* [1] proved some coupled fixed point results for contractive type conditions in partial metric spaces.

Recently, *Shukla* [22] generalized both the notions of b-metric and partial metric spaces by introducing partial b-metric spaces. He proved Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces. Also some examples are given which illustrate the results obtained in this new space.

The aim of this paper is to establish some coupled fixed point theorems for contractive type conditions in the setting of partial b-metric space. Our results of findings in this paper extend and generalize several previous findings from the existing literature.

In the sequel, first we recall some basic definitions from b-metric and partial metric spaces (see, [2, 14]).

Definition 1.1. Let $\mathcal{X}$ be a nonempty set and the mapping $d\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) satisfies:

($b\mathcal{M}1$) $d(\alpha,\beta) = 0$ if and only if $\alpha = \beta$ for all $\alpha,\beta \in \mathcal{X}$;

($b\mathcal{M}2$) $d(\alpha,\beta) = d(\beta,\alpha)$ for all $\alpha,\beta \in \mathcal{X}$;

($b\mathcal{M}3$) there exists a real number $s \geq 1$ such that $d(\alpha,\beta) \leq s[d(\alpha,\gamma) + d(\gamma,\beta)]$ for all $\alpha,\beta,\gamma \in \mathcal{X}$.

Then d is called a b-metric on $\mathcal{X}$ and the pair $(\mathcal{X}, d)$ is called a b-metric space with coefficient s.

Definition 1.2. A partial metric on a nonempty set $\mathcal{X}$ is a function $p\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ such that for all $\alpha,\beta,\gamma \in \mathcal{X}$:

($\mathcal{PM}1$) $\alpha = \beta$ if and only if $p(\alpha,\alpha) = p(\alpha,\beta) = p(\beta,\beta)$;

($\mathcal{PM}2$) $p(\alpha,\alpha) \leq p(\alpha,\beta)$;

($\mathcal{PM}3$) $p(\alpha,\beta) = p(\beta,\alpha)$;

($\mathcal{PM}4$) $p(\alpha,\beta) \leq p(\alpha,\gamma) + p(\gamma,\beta) - p(\gamma,\gamma)$.

A partial metric space is a pair $(\mathcal{X}, p)$ such that $\mathcal{X}$ is a nonempty set and p is a partial metric on $\mathcal{X}$.

In 2013, *S. Shukla* [22] define the following.

Definition 1.3. A partial b-metric on a nonempty set $\mathcal{X}$ is a function $p_b\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ such that for all $\alpha,\beta,\gamma \in \mathcal{X}$:

($\mathcal{P}b1$) $\alpha = \beta$ if and only if $p_b(\alpha,\alpha) = p_b(\alpha,\beta) = p_b(\beta,\beta)$;

($\mathcal{P}b2$) $p_b(\alpha,\alpha) \leq p_b(\alpha,\beta)$;

($\mathcal{P}b3$) $p_b(\alpha,\beta) = p_b(\beta,\alpha)$;

($\mathcal{P}b4$) there exists a real number $s \geq 1$ such that $p_b(\alpha,\beta) \leq s[p_b(\alpha,\gamma) + p_b(\gamma,\beta)] - p_b(\gamma,\gamma)$.

A partial b-metric space is a pair $(\mathcal{X}, p_b)$ such that $\mathcal{X}$ is a nonempty set and p_b is a partial b-metric on $\mathcal{X}$. The number s is called the coefficient of $(\mathcal{X}, p_b)$.

Remark 1.4. In a partial b-metric space $(\mathcal{X}, p_b)$ if $\alpha, \beta \in \mathcal{X}$ and $p_b(\alpha, \beta) = 0$, then $\alpha = \beta$, but the converse may not be true.

Remark 1.5. It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Example 1.6. ([22]) Let $\mathcal{X} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, $p > 1$ a constant and $p_b \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by

$$p_b(\alpha, \beta) = [\max\{\alpha, \beta\}]^p + |\alpha - \beta| \quad \text{for all } \alpha, \beta \in \mathcal{X}.$$

Then $(\mathbb{X}, p_b)$ is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space. Indeed, for any $\alpha > 0$, we have $p_b(\alpha, \alpha) = \alpha^p \neq 0$; therefore, p_b is not a b-metric on $\mathcal{X}$. Also, for $\alpha = 5$, $\beta = 1$, $\gamma = 4$ we have $p_b(\alpha, \beta) = 5^p + 4^p$ and $p_b(\alpha, \gamma) + p_b(\gamma, \beta) - p_b(\gamma, \gamma) = 5^p + 1 + 4^p + 3^p - 4^p = 5^p + 1 + 3^p$, so $p_b(\alpha, \beta) > p_b(\alpha, \gamma) + p_b(\gamma, \beta) - p_b(\gamma, \gamma)$ for all $p > 1$; therefore, p_b is not a partial metric on $\mathcal{X}$.

Every partial b-metric "p_b" on a nonempty set $\mathcal{X}$ generates a topology τ_{p_b} on $\mathcal{X}$ whose base is the family of open p_b-balls where $\tau_{p_b} = \{B_{p_b}(\alpha, \varepsilon) : \alpha \in \mathcal{X}, \varepsilon > 0\}$ and

$$B_{p_b}(\alpha, \varepsilon) = \{\beta \in \mathcal{X} : p_b(\alpha, \beta) < p_b(\alpha, \alpha) + \varepsilon\},$$

for all $\alpha \in \mathcal{X}$ and $\varepsilon > 0$. Obviously, the topological space $(\mathcal{X}, \tau_{p_b})$ is T_0, but need not be T_1.

Now we recall the definition of Cauchy sequence and convergent sequence in partial b-metric spaces.

Definition 1.7. ([22]) Let $(\mathcal{X}, p_b)$ be a partial b-metric space with coefficient s. Then:

- a sequence $\{q_n\}$ in $(\mathcal{X}, p_b)$ is said to be convergent with respect to τ_{p_b} and converges to a point $q \in \mathcal{X}$, if $\lim_{n\to\infty} p_b(q_n, q) = p_b(q, q)$;

- a sequence $\{q_n\}$ is said to be Cauchy sequence in $(\mathcal{X}, p_b)$ if $\lim_{n,m\to\infty} p_b(q_n, q_m)$ exists and is finite;

- $(\mathcal{X}, p_b)$ is said to be a complete partial b-metric space if for every Cauchy sequence $\{q_n\}$ in $\mathcal{X}$ there exists $q \in \mathcal{X}$ such that ,

$$\lim_{n,m\to\infty} p_b(q_n, q_m) = \lim_{n\to\infty} p_b(q_n, q) = p_b(q, q).$$

- A mapping $g \colon \mathcal{X} \to \mathcal{X}$ is said to be continuous at $q_0 \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $g\Big(B_{p_b}(q_0, \delta)\Big) \subset B_{p_b}\Big(g(q_0), \varepsilon\Big)$.

Note: In a partial b-metric space the limit of convergent sequence may not be unique.

Example 1.8. ([22]) Let $\mathcal{X} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, $c > 0$ be any constant and define $p_b \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ by

$$p_b(\alpha, \beta) = \max\{\alpha, \beta\} + c \quad \text{for all } \alpha, \beta \in \mathcal{X}.$$

Then $(\mathbb{X}, p_b)$ is a partial b-metric space with arbitrary coefficient $s \geq 1$. Now, define a sequence $\{\alpha_n\}$ in $\mathcal{X}$ by $\alpha_n = 1$ for all $n \in \mathbb{N}$. Note that, if $\beta \geq 1$, we have $p_b(\alpha_n, \beta) = \beta + c = p_b(\beta, \beta)$; therefore, $\lim_{n\to\infty} p_b(\alpha_n, \beta) = p_b(\beta, \beta)$ for all $\beta \geq 1$. Thus, the limit of convergent sequence in partial b-metric space need not be unique.

Definition 1.9. ([1]) An element $(\alpha, \beta) \in \mathcal{X} \times \mathcal{X}$ is said to be a coupled fixed point of the mapping $\psi \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ if $\psi(\alpha, \beta) = \alpha$ and $\psi(\beta, \alpha) = \beta$.

Example 1.10. Let $\mathcal{X} = [0, +\infty)$ and $\psi \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be defined by $\psi(\alpha, \beta) = \frac{\alpha+\beta}{6}$ for all $\alpha, \beta \in \mathcal{X}$. One can easily see that ψ has a unique coupled fixed point $(0, 0)$.

Example 1.11. Let $\mathcal{X} = [0, +\infty)$ and $\psi \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be defined by $\psi(\alpha, \beta) = \frac{\alpha+\beta}{2}$ for all $\alpha, \beta \in \mathcal{X}$. Then we see that ψ has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Motivated by the works of Shukla [22] and some others, the purpose of this paper is to establish some coupled fixed point results for contractive type conditions in the framework of partial b-metric spaces. The results of findings in this paper extend and generalize several previously published results from the existing literature (see, for example, [19], [1] and many others).

2. Main results

In this section, we shall prove some unique coupled fixed point theorems in the framework of partial b-metric spaces.

Theorem 2.1. *Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $\psi \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \beta, \eta, \theta \in \mathcal{X}$:*

$$\begin{aligned} p_b(\psi(\alpha, \beta), \psi(\eta, \theta)) \;\; \leq \;\; \frac{\lambda}{s^2} \max\Big\{ & p_b(\alpha, \eta), p_b(\psi(\alpha, \beta), \alpha), p_b(\psi(\eta, \theta), \eta), \\ & \frac{1}{2}[p_b(\psi(\alpha, \beta), \eta) + p_b(\psi(\alpha, \beta), \alpha)], \\ & \frac{p_b(\alpha, \eta)}{1 + p_b(\alpha, \eta)} \Big\}, \end{aligned} \tag{2.1}$$

where $s \geq 1$ and $\lambda \in (0, 1)$. Then ψ has a unique coupled fixed point in $\mathcal{X}$ and $p_b(a, a) = 0$ for some $a \in \mathcal{X}$.

Proof. Choose $\alpha_0, \beta_0 \in \mathcal{X}$. Set $\alpha_1 = \psi(\alpha_0, \beta_0)$ and $\beta_1 = \psi(\beta_0, \alpha_0)$. Repeating this process, we obtain two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathcal{X}$ such that $\alpha_{n+1} = \psi(\alpha_n, \beta_n)$

and $\beta_{n+1} = \psi(\beta_n, \alpha_n)$. Then, from equation (2.1) and using $(\mathcal{P}b2)$, $(\mathcal{P}b3)$, we have

$$\begin{aligned}
p_b(\alpha_n, \alpha_{n+1}) &= p_b(\psi(\alpha_{n-1}, \beta_{n-1}), \psi(\alpha_n, \beta_n)) \\
&\leq \frac{\lambda}{s^2} \max\Big\{p_b(\alpha_{n-1}, \alpha_n), p_b(\psi(\alpha_{n-1}, \beta_{n-1}), \alpha_{n-1}), p_b(\psi(\alpha_n, \beta_n), \alpha_n), \\
&\qquad \frac{1}{2}[p_b(\psi(\alpha_{n-1}, \beta_{n-1}), \alpha_n) + p_b(\psi(\alpha_{n-1}, \beta_{n-1}), \alpha_{n-1})], \\
&\qquad \frac{p_b(\alpha_{n-1}, \alpha_n)}{1 + p_b(\alpha_{n-1}, \alpha_n)}\Big\} \\
&= \frac{\lambda}{s^2} \max\Big\{p_b(\alpha_{n-1}, \alpha_n), p_b(\alpha_n, \alpha_{n-1}), p_b(\alpha_{n+1}, \alpha_n), \\
&\qquad \frac{1}{2}[p_b(\alpha_n, \alpha_n) + p_b(\alpha_n, \alpha_{n-1})], \frac{p_b(\alpha_{n-1}, \alpha_n)}{1 + p_b(\alpha_{n-1}, \alpha_n)}\Big\} \\
&= \frac{\lambda}{s^2} \max\Big\{p_b(\alpha_{n-1}, \alpha_n), p_b(\alpha_n, \alpha_{n+1})\Big\}. \qquad (2.2)
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
p_b(\beta_n, \beta_{n+1}) &= p_b(\psi(\beta_{n-1}, \alpha_{n-1}), \psi(\beta_n, \alpha_n)) \\
&\leq \frac{\lambda}{s^2} \max\Big\{p_b(\beta_{n-1}, \beta_n), p_b(\beta_n, \beta_{n+1})\Big\}. \qquad (2.3)
\end{aligned}$$

Set

$$\mathcal{P}_n = p_b(\alpha_n, \alpha_{n+1}) \text{ and } \mathcal{Q}_n = p_b(\beta_n, \beta_{n+1}). \qquad (2.4)$$

$$\Lambda_1 = \max\{\mathcal{P}_{n-1}, \mathcal{P}_n\}. \qquad (2.5)$$

and

$$\Lambda_2 = \max\{\mathcal{Q}_{n-1}, \mathcal{Q}_n\}. \qquad (2.6)$$

Now, we set

$$\begin{aligned}
\mathcal{W}_n &= \mathcal{P}_n + \mathcal{Q}_n \\
&= p_b(\alpha_n, \alpha_{n+1}) + p_b(\beta_n, \beta_{n+1}). \qquad (2.7)
\end{aligned}$$

Then from equations (2.2) - (2.7), we have

$$\mathcal{W}_n \leq \frac{\lambda}{s^2}\,(\Lambda_1 + \Lambda_2). \qquad (2.8)$$

If $\Lambda_1 = \mathcal{P}_n$ and $\Lambda_2 = \mathcal{Q}_n$, then from equation (2.8), we have

$$\begin{aligned}
\mathcal{W}_n &\leq \frac{\lambda}{s^2}\,(\mathcal{P}_n + \mathcal{Q}_n) \\
&= \frac{\lambda}{s^2}\,\mathcal{W}_n,
\end{aligned}$$

that is, $\Big(1 - \frac{\lambda}{s^2}\Big)\mathcal{W}_n \leq 0$, which is a contradiction, since $\lambda \in (0, 1)$ and $s \geq 1$.

Hence, we conclude that

$$\begin{aligned}\mathcal{W}_n &\leq \frac{\lambda}{s^2}(\mathcal{P}_{n-1}+\mathcal{Q}_{n-1}) \\ &= \frac{\lambda}{s^2}\mathcal{W}_{n-1}. \qquad (2.9)\end{aligned}$$

Then for each $n \in \mathbb{N}$, we have

$$\mathcal{W}_n \leq \frac{\lambda}{s^2}\mathcal{W}_{n-1} \leq \Big(\frac{\lambda}{s^2}\Big)^2\mathcal{W}_{n-2} \leq \cdots \leq \Big(\frac{\lambda}{s^2}\Big)^n\mathcal{W}_0. \qquad (2.10)$$

If $\mathcal{W}_0 = 0$, then $p_b(\alpha_0, \alpha_1) + p_b(\beta_0, \beta_1) = 0$. Hence, we get $\alpha_0 = \alpha_1 = \psi(\alpha_0, \beta_0)$ and $\beta_0 = \beta_1 = \psi(\beta_0, \alpha_0)$, that is, (α_0, β_0) is a coupled fixed point of ψ. Now, we assume that $\mathcal{W}_0 > 0$. For each $m \geq n$, where $n, m \in \mathbb{N}$, we have, by using condition $(\mathcal{P}b4)$

$$\begin{aligned}p_b(\alpha_n, \alpha_m) &\leq s[p_b(\alpha_n, \alpha_{n+1}) + p_b(\alpha_{n+1}, \alpha_m)] - p_b(\alpha_{n+1}, \alpha_{n+1}) \\ &\leq s\,p_b(\alpha_n, \alpha_{n+1}) + s^2[p_b(\alpha_{n+1}, \alpha_{n+2}) + p_b(\alpha_{n+2}, \alpha_m)] \\ &\quad -s\,p_b(\alpha_{n+2}, \alpha_{n+2}) \\ &\leq s\,p_b(\alpha_n, \alpha_{n+1}) + s^2 p_b(\alpha_{n+1}, \alpha_{n+2}) + s^3\,p_b(\alpha_{n+2}, \alpha_{n+3}) \\ &\quad + \cdots + s^{m-n}\,p_b(\alpha_{m-1}, \alpha_m). \qquad (2.11)\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}p_b(\beta_n, \beta_m) &\leq s\,p_b(\beta_n, \beta_{n+1}) + s^2 p_b(\beta_{n+1}, \beta_{n+2}) + s^3\,p_b(\beta_{n+2}, \beta_{n+3}) \\ &\quad + \cdots + s^{m-n}\,p_b(\beta_{m-1}, \beta_m). \qquad (2.12)\end{aligned}$$

Thus,

$$\begin{aligned}p_b(\alpha_n, \alpha_m) + p_b(\beta_n, \beta_m) &\leq s\mathcal{W}_n + s^2\mathcal{W}_{n+1} + \cdots + s^{m-n}\mathcal{W}_{m-1} \\ &\leq \Big[s\Big(\frac{\lambda}{s^2}\Big)^n + s^2\Big(\frac{\lambda}{s^2}\Big)^{n+1} + \cdots + s^{m-n}\Big(\frac{\lambda}{s^2}\Big)^{m-1}\Big]\mathcal{W}_0 \\ &= \Big[\sum_{i=1}^{m-n} s^i\Big(\frac{\lambda}{s^2}\Big)^{i+n-1}\Big]\mathcal{W}_0 = \Big[\sum_{t=1}^{m-1} s^t\Big(\frac{\lambda}{s^2}\Big)^t\Big]\mathcal{W}_0 \\ &\leq \Big[\sum_{t=1}^{n}\Big(\frac{\lambda}{s^2}\Big)^t\Big]\mathcal{W}_0 = \Big[\frac{\Big(\frac{\lambda}{s^2}\Big)^n}{1-\frac{\lambda}{s^2}}\Big]\mathcal{W}_0. \qquad (2.13)\end{aligned}$$

As $\lambda \in (0, \frac{1}{s})$ and $s \geq 1$, it follows from the inequality (2.13), we obtain that $p_b(\alpha_n, \alpha_m) + p_b(\beta_n, \beta_m) \to 0$ as $n \to \infty$, which implies that $\{\alpha_n\}$ and $\{\beta_n\}$ are Cauchy sequences in $(\mathcal{X}, p_b)$. Since the partial b-metric space $(\mathcal{X}, p_b)$ is complete, so there exist $c, d \in \mathcal{X}$ such that $\alpha_n \to c$, $\beta_n \to d$ as $n \to \infty$ and

$$p_b(c, c) = \lim_{n\to\infty} p_b(\alpha_n, c) = \lim_{n,m\to\infty} p_b(\alpha_n, \alpha_m) = 0, \qquad (2.14)$$

$$p_b(d, d) = \lim_{n\to\infty} p_b(\beta_n, d) = \lim_{n,m\to\infty} p_b(\beta_n, \beta_m) = 0. \qquad (2.15)$$

We now show that $c = \psi(c,d)$. Suppose on the contrary that $c \neq \psi(c,d)$ and $d \neq \psi(d,c)$ so that $0 < p_b(c,\psi(c,d)) = A_1$ and $0 < p_b(d,\psi(d,c)) = A_2$, then

$$\begin{aligned}
A_1 &= p_b(c,\psi(c,d)) \\
&\leq p_b(c,\alpha_{n+1}) + p_b(\alpha_{n+1},\psi(c,d)) - p_b(\alpha_{n+1},\alpha_{n+1}) \\
&\leq p_b(c,\alpha_{n+1}) + p_b(\alpha_{n+1},\psi(c,d)) \\
&= p_b(\alpha_{n+1},\psi(c,d)) + p_b(c,\alpha_{n+1}) \\
&= p_b(\psi(\alpha_n,\beta_n),\psi(c,d)) + p_b(c,\alpha_{n+1}) \\
&\leq \frac{\lambda}{s^2}\max\Big\{p_b(\alpha_n,c), p_b(\psi(\alpha_n,\beta_n),\alpha_n), p_b(\psi(c,d),c), \\
&\qquad \frac{1}{2}[p_b(\psi(\alpha_n,\beta_n),c) + p_b(\psi(\alpha_n,\beta_n),\alpha_n)], \\
&\qquad \frac{p_b(\alpha_n,c)}{1+p_b(\alpha_n,c)}\Big\} + p_b(c,\alpha_{n+1}) \\
&= \frac{\lambda}{s^2}\max\Big\{p_b(\alpha_n,c), p_b(\alpha_{n+1},\alpha_n), p_b(\psi(c,d),c), \\
&\qquad \frac{1}{2}[p_b(\alpha_{n+1},c) + p_b(\alpha_{n+1},\alpha_n)], \\
&\qquad \frac{p_b(\alpha_n,c)}{1+p_b(\alpha_n,c)}\Big\} + p_b(c,\alpha_{n+1}). \qquad (2.16)
\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (2.16) and using equation (2.14), we obtain $A_1 \leq \frac{\lambda}{s^2}\, p_b(c,\psi(c,d)) = \frac{\lambda}{s^2}\, A_1$, which is a contradiction, since $\lambda \in (0,1)$ and $s \geq 1$. Hence, we conclude that $A_1 = 0$, that is, $p_b(c,\psi(c,d)) = 0$ and so $\psi(c,d) = c$. Similarly, we can show that $\psi(d,c) = d$. This shows that (c,d) is a coupled fixed point of ψ.

Now, we show the uniqueness of the coupled fixed point. Suppose that (c',d') is another coupled fixed point of ψ such that $(c,d) \neq (c',d')$, then from equation (2.1) and using (2.14), (2.15) and $(\mathcal{P}b3)$, we have

$$\begin{aligned}
p_b(c,c') &= p_b(\psi(c,d),\psi(c',d')) \\
&\leq \frac{\lambda}{s^2}\max\Big\{p_b(c,c'), p_b(\psi(c,d),c), p_b(\psi(c',d'),c'), \\
&\qquad \frac{1}{2}[p_b(\psi(c,d),c') + p_b(\psi(c,d),c)], \frac{p_b(c,c')}{1+p_b(c,c')}\Big\} \\
&= \frac{\lambda}{s^2}\max\Big\{p_b(c,c'), p_b(c,c), p_b(c',c'), \\
&\qquad \frac{1}{2}[p_b(c,c') + p_b(c,c)], \frac{p_b(c,c')}{1+p_b(c,c')}\Big\} \\
&= \frac{\lambda}{s^2}\max\Big\{p_b(c,c'), 0, 0, \frac{1}{2}p_b(c,c'), \frac{p_b(c,c')}{1+p_b(c,c')}\Big\} \\
&= \frac{\lambda}{s^2}\, p_b(c,c'),
\end{aligned}$$

which is a contradiction, since $\lambda \in (0,1)$ and $s \geq 1$. Hence, we conclude that $p_b(c, c') = 0$, and so $c = c'$. Similarly, one can show that $d = d'$. This shows that the coupled fixed point of ψ is unique. This completes the proof. □

Theorem 2.2. *Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \beta, \eta, \theta \in \mathcal{X}$:*

$$\begin{aligned} p_b(\psi(\alpha,\beta), \psi(\eta,\theta)) \;\leq\; & k_1\, p_b(\alpha,\eta) + k_2\, p_b(\beta,\theta) + k_3\, p_b(\psi(\alpha,\beta),\alpha) \\ & + k_4\, p_b(\psi(\eta,\theta),\eta) + k_5\, p_b(\psi(\alpha,\beta),\eta) \\ & + k_6\, p_b(\psi(\eta,\theta),\alpha), \end{aligned} \tag{2.17}$$

where $k_1, k_2, \ldots, k_6$ are nonnegative constants such that $k_1 + k_2 + k_3 + k_4 + k_5 + 2sk_6 < 1$. Then ψ has a unique coupled fixed point in $\mathcal{X}$.

Proof. Choose $\alpha_0, \beta_0 \in \mathcal{X}$. Set $\alpha_1 = \psi(\alpha_0, \beta_0)$ and $\beta_1 = \psi(\beta_0, \alpha_0)$. Repeating this process, we obtain two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathcal{X}$ such that $\alpha_{n+1} = \psi(\alpha_n, \beta_n)$ and $\beta_{n+1} = \psi(\beta_n, \alpha_n)$. Set $\rho_n = p_b(\alpha_n, \alpha_{n+1})$ and $\sigma_n = p_b(\beta_n, \beta_{n+1})$. Then, from equations (2.17) and using $(\mathcal{P}b2)$, $(\mathcal{P}b3)$, $(\mathcal{P}b4)$, we have

$$\begin{aligned} p_b(\alpha_n, \alpha_{n+1}) &= p_b(\psi(\alpha_{n-1},\beta_{n-1}), \psi(\alpha_n,\beta_n)) \\ &\leq k_1\, p_b(\alpha_{n-1},\alpha_n) + k_2\, p_b(\beta_{n-1},\beta_n) + k_3\, p_b(\psi(\alpha_{n-1},\beta_{n-1}),\alpha_{n-1}) \\ &\quad + k_4\, p_b(\psi(\alpha_n,\beta_n),\alpha_n) + k_5\, p_b(\psi(\alpha_{n-1},\beta_{n-1}),\alpha_n) \\ &\quad + k_6\, p_b(\psi(\alpha_n,\beta_n),\alpha_{n-1}) \\ &= k_1\, p_b(\alpha_{n-1},\alpha_n) + k_2\, p_b(\beta_{n-1},\beta_n) + k_3\, p_b(\alpha_n,\alpha_{n-1}) \\ &\quad + k_4\, p_b(\alpha_{n+1},\alpha_n) + k_5\, p_b(\alpha_n,\alpha_n) + k_6\, p_b(\alpha_{n+1},\alpha_{n-1}) \\ &\leq k_1\, p_b(\alpha_{n-1},\alpha_n) + k_2\, p_b(\beta_{n-1},\beta_n) + k_3\, p_b(\alpha_n,\alpha_{n-1}) \\ &\quad + k_4\, p_b(\alpha_{n+1},\alpha_n) + k_5\, p_b(\alpha_n,\alpha_{n+1}) + k_6 s\, [p_b(\alpha_{n+1},\alpha_n) \\ &\quad + p_b(\alpha_n,\alpha_{n-1})] - k_6\, p_b(\alpha_n,\alpha_n) \\ &\leq k_1\, p_b(\alpha_{n-1},\alpha_n) + k_2\, p_b(\beta_{n-1},\beta_n) + k_3\, p_b(\alpha_n,\alpha_{n-1}) \\ &\quad + k_4\, p_b(\alpha_{n+1},\alpha_n) + k_5\, p_b(\alpha_n,\alpha_{n+1}) + sk_6\, [p_b(\alpha_{n+1},\alpha_n) \\ &\quad + p_b(\alpha_n,\alpha_{n-1})] \\ &= (k_1 + k_3 + sk_6)\, p_b(\alpha_{n-1},\alpha_n) + (k_4 + k_5 + sk_6)\, p_b(\alpha_n,\alpha_{n+1}) \\ &\quad + k_2\, p_b(\beta_{n-1},\beta_n), \end{aligned}$$

that is,

$$\rho_n \leq (k_1 + k_3 + sk_6)\, \rho_{n-1} + (k_4 + k_5 + sk_6)\, \rho_n + k_2\, \sigma_{n-1}. \tag{2.18}$$

Similarly, we obtain

$$\begin{aligned} p_b(\beta_n, \beta_{n+1}) &= p_b(\psi(\beta_{n-1},\alpha_{n-1}), \psi(\beta_n,\alpha_n)) \\ &\leq (k_1 + k_3 + sk_6)\, p_b(\beta_{n-1},\beta_n) + (k_4 + k_5 + sk_6)\, p_b(\beta_n,\beta_{n+1}) \\ &\quad + k_2\, p_b(\alpha_{n-1},\alpha_n), \end{aligned}$$

that is,

$$\sigma_n \leq (k_1 + k_3 + sk_6)\, \sigma_{n-1} + (k_4 + k_5 + sk_6)\, \sigma_n + k_2\, \rho_{n-1}. \tag{2.19}$$

Now, we set

$$\tau_n = \rho_n + \sigma_n. \tag{2.20}$$

Then from equations (2.18)-(2.20), we obtain

$$\begin{aligned}
\tau_n &= \rho_n + \sigma_n \\
&\le (k_1 + k_3 + sk_6)\,[\rho_{n-1} + \sigma_{n-1}] + (k_4 + k_5 + sk_6)\,[\rho_n + \sigma_n] \\
&\quad + k_2\,[\rho_{n-1} + \sigma_{n-1}] \\
&= (k_1 + k_3 + sk_6)\,\tau_{n-1} + (k_4 + k_5 + sk_6)\,\tau_n + k_2\,\tau_{n-1} \\
&= (k_1 + k_2 + k_3 + sk_6)\,\tau_{n-1} + (k_4 + k_5 + sk_6)\,\tau_n,
\end{aligned}$$

which implies

$$\begin{aligned}
\tau_n &\le \Big(\frac{k_1 + k_2 + k_3 + sk_6}{1 - k_4 - k_5 - sk_6}\Big)\,\tau_{n-1} \\
&= \Delta\,\tau_{n-1},
\end{aligned} \tag{2.21}$$

where $\Delta = \Big(\frac{k_1+k_2+k_3+sk_6}{1-k_4-k_5-sk_6}\Big) < 1$, since $k_1 + k_2 + k_3 + k_4 + k_5 + 2sk_6 < 1$.

Then for each $n \in \mathbb{N}$, we have

$$\tau_n \le \Delta\tau_{n-1} \le \Delta^2\tau_{n-2} \le \cdots \le \Delta^n\tau_0. \tag{2.22}$$

If $\tau_0 = 0$, then $p_b(\alpha_0, \alpha_1) + p_b(\beta_0, \beta_1) = 0$. Hence, we get $\alpha_0 = \alpha_1 = \psi(\alpha_0, \beta_0)$ and $\beta_0 = \beta_1 = \psi(\beta_0, \alpha_0)$, that is, (α_0, β_0) is a coupled fixed point of ψ. Now, we assume that $\tau_0 > 0$. For each $m \ge n$, where $n, m \in \mathbb{N}$, we have, by using condition $(\mathcal{P}b4)$

$$\begin{aligned}
p_b(\alpha_n, \alpha_m) &\le s[p_b(\alpha_n, \alpha_{n+1}) + p_b(\alpha_{n+1}, \alpha_m)] - p_b(\alpha_{n+1}, \alpha_{n+1}) \\
&\le s\,p_b(\alpha_n, \alpha_{n+1}) + s^2[p_b(\alpha_{n+1}, \alpha_{n+2}) + p_b(\alpha_{n+2}, \alpha_m)] \\
&\quad - s\,p_b(\alpha_{n+2}, \alpha_{n+2}) \\
&\le s\,p_b(\alpha_n, \alpha_{n+1}) + s^2 p_b(\alpha_{n+1}, \alpha_{n+2}) + s^3\,p_b(\alpha_{n+2}, \alpha_{n+3}) \\
&\quad + \cdots + s^{m-n}\,p_b(\alpha_{m-1}, \alpha_m).
\end{aligned} \tag{2.23}$$

Similarly, we obtain

$$\begin{aligned}
p_b(\beta_n, \beta_m) &\le s\,p_b(\beta_n, \beta_{n+1}) + s^2 p_b(\beta_{n+1}, \beta_{n+2}) + s^3\,p_b(\beta_{n+2}, \beta_{n+3}) \\
&\quad + \cdots + s^{m-n}\,p_b(\beta_{m-1}, \beta_m).
\end{aligned} \tag{2.24}$$

Thus,

$$\begin{aligned}
p_b(\alpha_n,\alpha_m)+p_b(\beta_n,\beta_m) &\leq s\tau_n+s^2\tau_{n+1}+\cdots+s^{m-n}\tau_{m-1}\\
&\leq \Big[s\Delta^n+s^2\Delta^{n+1}+\cdots+s^{m-n}\Delta^{m-1}\Big]\tau_0\\
&= \Big[\sum_{r=1}^{m-n}s^r\Delta^{r+n-1}\Big]\tau_0\\
&= \Big[\sum_{p=1}^{m-1}s^p\Delta^p\Big]\tau_0\leq\Big[\sum_{p=1}^{n}s^p\Delta^p\Big]\tau_0\\
&= \Big[\frac{(s\Delta)^n}{1-s\Delta}\Big]\tau_0. \qquad (2.25)
\end{aligned}$$

As $\Delta\in(0,1)$ and $s\geq 1$, it follows from the inequality (2.25), we obtain that $p_b(\alpha_n,\alpha_m)+p_b(\beta_n,\beta_m)\to 0$ as $n\to\infty$, which implies that $\{\alpha_n\}$ and $\{\beta_n\}$ are Cauchy sequences in $(\mathcal{X},p_b)$. Since the partial b-metric space $(\mathcal{X},p_b)$ is complete, so there exist $j,k\in\mathcal{X}$ such that $\alpha_n\to j$, $\beta_n\to k$ as $n\to\infty$ and

$$p_b(j,j)=\lim_{n\to\infty}p_b(\alpha_n,j)=\lim_{n,m\to\infty}p_b(\alpha_n,\alpha_m)=0, \qquad (2.26)$$

$$p_b(k,k)=\lim_{n\to\infty}p_b(\beta_n,k)=\lim_{n,m\to\infty}p_b(\beta_n,\beta_m)=0. \qquad (2.27)$$

We now show that $j=\psi(j,k)$. Suppose on the contrary that $j\neq\psi(j,k)$ and $k\neq\psi(k,j)$ so that $0<p_b(j,\psi(j,k))=L_1$ and $0<p_b(k,\psi(k,j))=L_2$, then

$$\begin{aligned}
L_1 &= p_b(j,\psi(j,k))\\
&\leq p_b(j,\alpha_{n+1})+p_b(\alpha_{n+1},\psi(j,k))\\
&\quad -p_b(\alpha_{n+1},\alpha_{n+1})\\
&\leq p_b(j,\alpha_{n+1})+p_b(\alpha_{n+1},\psi(j,k))\\
&= p_b(\alpha_{n+1},\psi(j,k))+p_b(j,\alpha_{n+1})\\
&= p_b(\psi(\alpha_n,\beta_n),\psi(j,k))+p_b(j,\alpha_{n+1})\\
&\leq k_1\,p_b(\alpha_n,j)+k_2\,p_b(\beta_n,k)+k_3\,p_b(\psi(\alpha_n,\beta_n),\alpha_n)\\
&\quad +k_4\,p_b(\psi(j,k),j)+k_5\,p_b(\psi(\alpha_n,\beta_n),j)\\
&\quad +k_6\,p_b(\psi(j,k),\alpha_n)+p_b(j,\alpha_{n+1})\\
&= k_1\,p_b(\alpha_n,j)+k_2\,p_b(\beta_n,k)+k_3\,p_b(\alpha_{n+1},\alpha_n)\\
&\quad +k_4\,p_b(\psi(j,k),j)+k_5\,p_b(\alpha_{n+1},j)\\
&\quad +k_6\,p_b(\psi(j,k),\alpha_n)+p_b(j,\alpha_{n+1}). \qquad (2.28)
\end{aligned}$$

Passing to the limit as $n\to\infty$ in equation (2.28) and using equation (2.26) and $(\mathcal{P}b3)$, we obtain

$$L_1\leq(k_4+k_6)\,p_b(j,\psi(j,k))=(k_4+k_6)\,L_1,$$

which is a contradiction, since $k_4+k_6<1$. Hence, we conclude that $L_1=0$, that is, $p_b(j,\psi(j,k))=0$ and so $\psi(j,k)=j$. Similarly, we can show that $\psi(k,j)=k$. This shows that (j,k) is a coupled fixed point of ψ.

Now, we show the uniqueness of the coupled fixed point of ψ. Suppose that (j', k') is another coupled fixed point of ψ such that $(j, k) \neq (j', k')$, then from equation (2.17) and using (2.26), (2.27) and ($\mathcal{P}b3$), we have

$$\begin{aligned}
p_b(j, j') &= p_b(\psi(j, k), \psi(j', k')) \\
&\leq k_1\, p_b(j, j') + k_2\, p_b(k, k') + k_3\, p_b(\psi(j, k), j) \\
&\quad + k_4\, p_b(\psi(j', k'), j') + k_5\, p_b(\psi(j, k), j') \\
&\quad + k_6\, p_b(\psi(j', k'), j) \\
&= k_1\, p_b(j, j') + k_2\, p_b(k, k') + k_3\, p_b(j, j) \\
&\quad + k_4\, p_b(j', j') + k_5\, p_b(j, j') \\
&\quad + k_6\, p_b(j', j) \\
&= (k_1 + k_5 + k_6)\, p_b(j, j') + k_2\, p_b(k, k'). \qquad (2.29)
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
p_b(k, k') &= p_b(\psi(k, j), \psi(k', j')) \\
&\leq (k_1 + k_5 + k_6)\, p_b(k, k') + k_2\, p_b(j, j'). \qquad (2.30)
\end{aligned}$$

Hence from equations (2.29) and (2.30), we obtain

$$\begin{aligned}
p_b(j, j') + p_b(k, k') &\leq (k_1 + k_5 + k_6)\, [p_b(j, j') + p_b(k, k')] \\
&\quad + k_2\, [p_b(j, j') + p_b(k, k')] \\
&= (k_1 + k_2 + k_5 + k_6)\, [p_b(j, j') + p_b(k, k')],
\end{aligned}$$

which is a contradiction, since $k_1 + k_2 + k_5 + k_6 < 1$. Hence, we conclude that $p_b(j, j') + p_b(k, k') = 0$, and so $j = j'$ and $k = k'$. This shows that the coupled fixed point of ψ is unique. This completes the proof. $\square$

If we set $k_1 = k$, $k_2 = l$ and $k_3 = k_4 = k_5 = k_6 = 0$ in Theorem 2.2, then we have the following result.

Corollary 2.3. *Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \beta, \eta, \theta \in \mathcal{X}$:*

$$p_b(\psi(\alpha, \beta), \psi(\eta, \theta)) \leq k\, p_b(\alpha, \eta) + l\, p_b(\beta, \theta),$$

where k, l are nonnegative constants such that $k + l < 1$. Then ψ has a unique coupled fixed point in $\mathcal{X}$.

If we set $k_3 = k$, $k_4 = l$ and $k_1 = k_2 = k_5 = k_6 = 0$ in Theorem 2.2, then we have the following result.

Corollary 2.4. *Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \beta, \eta, \theta \in \mathcal{X}$:*

$$p_b(\psi(\alpha, \beta), \psi(\eta, \theta)) \leq k\, p_b(\psi(\alpha, \beta), \alpha) + l\, p_b(\psi(\eta, \theta), \eta),$$

where k, l are nonnegative constants such that $k + l < 1$. Then ψ has a unique coupled fixed point in $\mathcal{X}$.

If we set $k_5 = k$, $k_6 = l$ and $k_1 = k_2 = k_3 = k_4 = 0$ in Theorem 2.2, then we have the following result.

Corollary 2.5. *Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \beta, \eta, \theta \in \mathcal{X}$:*

$$p_b(\psi(\alpha,\beta),\psi(\eta,\theta)) \leq k\, p_b(\psi(\alpha,\beta),\eta) + l\, p_b(\psi(\eta,\theta),\alpha),$$

where k, l are nonnegative constants such that $k + 2sl < 1$. Then ψ has a unique coupled fixed point in $\mathcal{X}$.

Remark 2.6. (1) Theorem 2.1 and 2.2 extend and generalize the results of Aydi [1] from partial metric space to the setting of partial b-metric spaces.

(2) Theorem 2.1 and 2.2 also extend and generalize the results of Sabetghadam *et al.* [19] from cone metric space to the setting of partial b-metric spaces.

(3) Corollary 2.3, Corollary 2.4 and Corollary 2.5 extend and generalize Theorem 2.1, Theorem 2.4 and Theorem 2.5 of Aydi [1] from partial metric spaces to the setting of partial b-metric spaces.

Example 2.7. Let $\mathcal{X} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, $p > 1$ a constant and $p_b\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by

$$p_b(\alpha,\beta) = [\max\{\alpha,\beta\}]^p + |\alpha - \beta| \quad \text{for all } \alpha, \beta \in \mathcal{X}.$$

Then $(\mathbb{X}, p_b)$ is a partial b-metric space with coefficient $s = 2^p > 1$. Consider the mapping $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by $\psi(\alpha,\beta) = \frac{\alpha+\beta}{6}$. Now, for any $\alpha, \beta, \eta, \theta \in \mathcal{X}$, we have

$$\begin{aligned}
p_b(\psi(\alpha,\beta),\psi(\eta,\theta)) &= [\max\{\psi(\alpha,\beta),\psi(\eta,\theta)\}]^p + |\psi(\alpha,\beta) - \psi(\eta,\theta)| \\
&= \Big[\max\Big\{\frac{\alpha+\beta}{6}, \frac{\eta+\theta}{6}\Big\}\Big]^p + \Big|\frac{\alpha+\beta}{6} - \frac{\eta+\theta}{6}\Big| \\
&= \Big[\frac{1}{6}\max\{\alpha+\beta, \eta+\theta\}\Big]^p + \frac{1}{6}|(\alpha+\beta) - (\eta+\theta)| \\
&\leq \Big[\frac{1}{6}\max\{\alpha,\eta\} + \frac{1}{6}\max\{\beta,\theta\}\Big]^p + \frac{1}{6}|(\alpha-\eta) + (\beta-\theta)| \\
&\leq \Big[\frac{1}{6}\max\{\alpha,\eta\}\Big]^p + \Big[\frac{1}{6}\max\{\beta,\theta\}\Big]^p + \frac{1}{6}|\alpha-\eta| + \frac{1}{6}|\beta-\theta| \\
&\leq \frac{1}{6}\Big([\max\{\alpha,\eta\}]^p + |\alpha-\eta|\Big) + \frac{1}{6}\Big([\max\{\beta,\theta\}]^p + |\beta-\theta|\Big) \\
&= k\, p_b(\alpha,\eta) + l\, p_b(\beta,\theta),
\end{aligned}$$

which is the contractive condition of Corollary 2.3 for $k = l = \frac{1}{6}$ with $k+l = \frac{1}{3} < 1$. Therefore, by Corollary 2.3, ψ has a unique coupled fixed point which is $(0,0)$ in $\mathcal{X}$. Note that if the mapping $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is given by $\psi(\alpha,\beta) = \frac{\alpha+\beta}{2}$, then ψ

satisfies contractive condition of Corollary 2.3 for $k = l = 1/2$, that is,

$$\begin{aligned}
p_b(\psi(\alpha,\beta),\psi(\eta,\theta)) &= [\max\{\psi(\alpha,\beta),\psi(\eta,\theta)\}]^p + |\psi(\alpha,\beta)-\psi(\eta,\theta)| \\
&= \Big[\max\Big\{\frac{\alpha+\beta}{2},\frac{\eta+\theta}{2}\Big\}\Big]^p + \Big|\frac{\alpha+\beta}{2}-\frac{\eta+\theta}{2}\Big| \\
&= \Big[\frac{1}{2}\max\{\alpha+\beta,\eta+\theta\}\Big]^p + \frac{1}{2}|(\alpha+\beta)-(\eta+\theta)| \\
&\le \Big[\frac{1}{2}\max\{\alpha,\eta\}+\frac{1}{2}\max\{\beta,\theta\}\Big]^p + \frac{1}{2}|(\alpha-\eta)+(\beta-\theta)| \\
&\le \Big[\frac{1}{2}\max\{\alpha,\eta\}\Big]^p + \Big[\frac{1}{2}\max\{\beta,\theta\}\Big]^p + \frac{1}{2}|\alpha-\eta| + \frac{1}{2}|\beta-\theta| \\
&\le \frac{1}{2}\Big([\max\{\alpha,\eta\}]^p + |\alpha-\eta|\Big) + \frac{1}{2}\Big([\max\{\beta,\theta\}]^p + |\beta-\theta|\Big) \\
&= k\, p_b(\alpha,\eta) + l\, p_b(\beta,\theta),
\end{aligned}$$

In this case $(0,0)$ and $(1,1)$ are both coupled fixed points of ψ, and hence, the coupled fixed point of ψ is not unique. This shows that the condition $k+l<1$ in Corollary 2.3 cannot be omitted in the statement of the aforesaid result. By similar way, we can verify the results of Corollary 2.4 and Corollary 2.5.

We define as $\mathcal{T}(\alpha) = \psi(\alpha,\alpha)$ for all $\alpha \in \mathcal{X}$. Then, we have the following results.

Corollary 2.8. *([22], Theorem 1) Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \ge 1$. Suppose that the mapping $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \eta \in \mathcal{X}$:*

$$p_b(\mathcal{T}(\alpha),\mathcal{T}(\eta)) \le k\, p_b(\alpha,\eta),$$

where $k \in [0,1)$ is a constant. Then $\mathcal{T}$ has a unique fixed point $u \in \mathcal{X}$ and $p_b(u,u) = 0$.

Proof. Taking $\alpha = \beta$, $\eta = \theta$, $k_1 = k$ and $k_2 = k_3 = k_4 = k_5 = k_6 = 0$ in Theorem 2.2. Then, we have the conclusion of the corollary from Theorem 2.2. □

Corollary 2.9. *([22], Theorem 2) Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \ge 1$. Suppose that the mapping $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \eta \in \mathcal{X}$:*

$$p_b(\mathcal{T}(\alpha),\mathcal{T}(\eta)) \le \lambda\, [p_b(\mathcal{T}(\alpha),\alpha) + p_b(\mathcal{T}(\eta),\eta)],$$

where $\lambda \in [0,\frac{1}{2})$, $\lambda \ne \frac{1}{s}$ is a constant. Then $\mathcal{T}$ has a unique fixed point $u \in \mathcal{X}$ and $p_b(u,u) = 0$.

Proof. Taking $\alpha = \beta$, $\eta = \theta$, $k_3 = k_4 = \lambda$ and $k_1 = k_2 = k_5 = k_6 = 0$ in Theorem 2.2. Then, we have the conclusion of the corollary from Theorem 2.2. □

We also obtain the following special case from Theorem 2.2.

Corollary 2.10. *([22], Theorem 3) Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s > 1$. Suppose that the mapping $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $\alpha, \eta \in \mathcal{X}$:*

$$p_b(\mathcal{T}(\alpha),\mathcal{T}(\eta)) \le \lambda \max\Big\{p_b(\alpha,\eta), p_b(\mathcal{T}(\alpha),\alpha), p_b(\mathcal{T}(\eta),\eta)\Big\},$$

where $\lambda \in [0, \frac{1}{s})$ *is a constant. Then* $\mathcal{T}$ *has a unique fixed point* $u \in \mathcal{X}$ *and* $p_b(u,u) = 0$.

Example 2.11. Let $\mathcal{X} = \{1,2,3,4\}$ and $p_b\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be defined by

$$p_b(\alpha,\eta) = \begin{cases} |\alpha-\eta|^2 + \max\{\alpha,\eta\}, & \text{if } \alpha \neq \eta, \\ \alpha, & \text{if } \alpha = \eta \neq 1, \\ 0, & \text{if } \alpha = \eta = 1, \end{cases}$$

for all $\alpha, \eta \in \mathcal{X}$. Then $(\mathcal{X}, p_b)$ is a complete partial b-metric space with the coefficient $s = 4 > 1$.

Define the mapping $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}(1) = 1,\ \mathcal{T}(2) = 1,\ \mathcal{T}(3) = 2,\ \mathcal{T}(4) = 2.$$

Now, we have

$$p_b(\mathcal{T}(1), \mathcal{T}(2)) = p_b(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p_b(1,2),$$

$$p_b(\mathcal{T}(1), \mathcal{T}(3)) = p_b(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p_b(1,3),$$

$$p_b(\mathcal{T}(1), \mathcal{T}(4)) = p_b(1,2) = 3 \leq \frac{3}{4}.13 = \frac{3}{4}p_b(1,4),$$

$$p_b(\mathcal{T}(2), \mathcal{T}(3)) = p_b(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p_b(2,3),$$

$$p_b(\mathcal{T}(2), \mathcal{T}(4)) = p_b(1,2) = 3 \leq \frac{3}{4}.8 = \frac{3}{4}p_b(2,4),$$

$$p_b(\mathcal{T}(3), \mathcal{T}(4)) = p_b(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p_b(3,4).$$

Thus, $\mathcal{T}$ satisfies all the conditions of Corollary 2.8 with $k = \frac{3}{4} < 1$. Now by applying Corollary 2.8, $\mathcal{T}$ has a unique fixed point namely 1 is the unique fixed point of $\mathcal{T}$. We note that, since $p_b(2,2) = 2 \neq 0$ it follows that p_b is not a b-metric. Also p_b is not a partial metric. Indeed, $p_b(4,1) = 13 > 9 = p_b(4,3) + p_b(3,1) - p_b(3,3)$. Therefor, results from [6] and [14] are not applicable while Corollary 2.8 is applicable.

3. Well-Posedness Theorem

In this section, we deal with the well-posedness of coupled fixed point problem studied in the previous section.

Definition 3.1. ([18]) Let $(\mathcal{X}, d)$ be a metric space and let $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ be a mapping. The problem of finding a fixed point of $\mathcal{T}$ is said to be well posed if:

(1) $\mathcal{T}$ has a unique fixed point α_0,

(2) for any sequence $\{\alpha_n\} \in \mathcal{X}$ with $\lim_{n\to\infty} d(\mathcal{T}\alpha_n, \alpha_n) = 0$, we have $\lim_{n\to\infty} d(\alpha_n, \alpha_0) = 0$.

Now, we define well-posedness of coupled fixed point in partial b-metric spaces.

Let $\mathcal{C}_0\mathcal{FP}(\psi, \mathcal{X} \times \mathcal{X})$ denote a coupled fixed point problem of mapping ψ and let $\mathcal{C}_0\mathcal{F}(\psi)$ denote the set of all coupled fixed points of ψ.

Definition 3.2. Let $(\mathcal{X}, p_b)$ be a partial b-metric space with the coefficient $s \geq 1$ and let $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a mapping. $\mathcal{C}_0\mathcal{FP}(\psi, \mathcal{X} \times \mathcal{X})$ is called well posed if:
(1) $\mathcal{C}_0\mathcal{F}(\psi)$ is unique,
(2) for any sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $\mathcal{X}$ with $(\bar{\alpha}, \bar{\beta}) \in \mathcal{C}_0\mathcal{F}(\psi)$ and

$$\lim_{n\to\infty} p_b(\psi(\alpha_n, \beta_n), \alpha_n) = 0 = \lim_{n\to\infty} p_b(\psi(\beta_n, \alpha_n), \beta_n)$$

implies

$$\bar{\alpha} = \lim_{n\to\infty} \alpha_n, \;\; \bar{\beta} = \lim_{n\to\infty} \beta_n.$$

Theorem 3.3. *Let $(\mathcal{X}, p_b)$ be a complete partial b-metric space with the coefficient $s \geq 1$ and $\psi\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be a mapping as in Theorem 2.2. For any sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $\mathcal{X}$ and $(\alpha, \beta) \in \mathcal{C}_0\mathcal{F}(\psi)$, if*

$$\lim_{n\to\infty} p_b(\alpha, \psi(\alpha_n, \beta_n)) = 0 = \lim_{n\to\infty} p_b(\beta, \psi(\beta_n, \alpha_n)),$$

then the coupled fixed point problem of ψ is well-posed with $p_b(x, x) = 0$ for some $x \in \mathcal{X}$.

Proof. From Theorem 2.2, the mapping ψ has a unique coupled fixed point, $(\alpha_0, \beta_0) \in \mathcal{X} \times \mathcal{X}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ in $\mathcal{X}$ and

$$\lim_{n\to\infty} p_b(\psi(\alpha_n, \beta_n), \alpha_n) = 0 = \lim_{n\to\infty} p_b(\psi(\beta_n, \alpha_n), \beta_n).$$

Without loss of generality, we assume that $(\alpha_0, \beta_0) \neq (\alpha_n, \beta_n)$ for any non-negative integer n. Using $\psi(\alpha_0, \beta_0) = \alpha_0$ and $\psi(\beta_0, \alpha_0) = \beta_0$, we obtain

$$\begin{aligned}
p_b(\alpha_0, \alpha_n) &\leq p_b(\psi(\alpha_0, \beta_0), \psi(\alpha_n, \beta_n)) + p_b(\psi(\alpha_n, \beta_n), \alpha_n) \\
&\quad - P_b(\psi(\alpha_n, \beta_n), \psi(\alpha_n, \beta_n)) \\
&\leq p_b(\psi(\alpha_0, \beta_0), \psi(\alpha_n, \beta_n)) + p_b(\psi(\alpha_n, \beta_n), \alpha_n) \\
&\leq k_1\, p_b(\alpha_0, \alpha_n) + k_2\, p_b(\beta_0, \beta_n) + k_3\, p_b(\psi(\alpha_0, \beta_0), \alpha_0) \\
&\quad + k_4\, p_b(\psi(\alpha_n, \beta_n), \alpha_n) + k_5\, p_b(\psi(\alpha_0, \beta_0), \alpha_n) \\
&\quad + k_6\, p_b(\psi(\alpha_n, \beta_n), \alpha_0) + p_b(\psi(\alpha_n, \beta_n), \alpha_n), \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
p_b(\beta_0, \beta_n) &\leq p_b(\psi(\beta_0, \alpha_0), \psi(\beta_n, \alpha_n)) + p_b(\psi(\beta_n, \alpha_n), \beta_n) \\
&\quad - P_b(\psi(\beta_n, \alpha_n), \psi(\beta_n, \alpha_n)) \\
&\leq p_b(\psi(\beta_0, \alpha_0), \psi(\beta_n, \alpha_n)) + p_b(\psi(\beta_n, \alpha_n), \beta_n) \\
&\leq k_1\, p_b(\beta_0, \beta_n) + k_2\, p_b(\alpha_0, \alpha_n) + k_3\, p_b(\psi(\beta_0, \alpha_0), \beta_0) \\
&\quad + k_4\, p_b(\psi(\beta_n, \alpha_n), \beta_n) + k_5\, p_b(\psi(\beta_0, \alpha_0), \beta_n) \\
&\quad + k_6\, p_b(\psi(\beta_n, \alpha_n), \beta_0) + p_b(\psi(\beta_n, \alpha_n), \beta_n). \qquad (3.2)
\end{aligned}$$

Since

$$\lim_{n\to\infty} p_b(\alpha_0, \psi(\alpha_n, \beta_n)) = 0 = \lim_{n\to\infty} p_b(\beta_0, \psi(\beta_n, \alpha_n)),$$

for $(\alpha_0, \beta_0) \in \mathcal{C}_0\mathcal{F}(\psi)$, using $(\mathcal{P}b3)$, we obtain

$$\lim_{n\to\infty} p_b(\alpha_0, \alpha_n) \leq (k_1 + k_5) \lim_{n\to\infty} p_b(\alpha_0, \alpha_n) + k_2 \lim_{n\to\infty} p_b(\beta_0, \beta_n), \qquad (3.3)$$

and

$$\lim_{n\to\infty} p_b(\beta_0,\beta_n) \le (k_1+k_5) \lim_{n\to\infty} p_b(\beta_0,\beta_n) + k_2 \lim_{n\to\infty} p_b(\alpha_0,\alpha_n). \tag{3.4}$$

Set

$$S_1 = p_b(\alpha_0,\alpha_n) \quad and \quad S_2 = p_b(\beta_0,\beta_n). \tag{3.5}$$

Now from equations (3.3)-(3.5), we have

$$\begin{aligned}\lim_{n\to\infty} S_1 + \lim_{n\to\infty} S_2 &\le (k_1+k_5)\,[\lim_{n\to\infty} S_1 + \lim_{n\to\infty} S_2] \\ &\quad + k_2\,[\lim_{n\to\infty} S_1 + \lim_{n\to\infty} S_2] \\ &= (k_1+k_2+k_5)\,[\lim_{n\to\infty} S_1 + \lim_{n\to\infty} S_2] \\ &< \lim_{n\to\infty} S_1 + \lim_{n\to\infty} S_2,\end{aligned}$$

which is a contradiction, since $k_1+k_2+k_5 < 1$. Hence, $\lim_{n\to\infty} S_1 + \lim_{n\to\infty} S_2 = 0$, that is, $\lim_{n\to\infty} p_b(\alpha_0,\alpha_n) + \lim_{n\to\infty} p_b(\beta_0,\beta_n) = 0$ and hence

$$\lim_{n\to\infty} \alpha_n = \alpha_0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \beta_0.$$

Hence the coupled fixed point problem of the mapping is well-posed. This completes the proof. □

4. Conclusion

In this paper, we establish some unique coupled fixed point results in the setting of partial b-metric spaces and give some corollaries of the established results. Further, we provide an example to validate the result. We also prove well-posedness of a coupled fixed point problem. Our results extend and generalize several previously published results from the existing literature (see, for example, [1, 19] and many others).

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References

1. H. Aydi, *Some coupled fixed point results on partial metric spaces*, International J. Math. Math. Sci. 2011, Article ID 647091, 11 pages.
2. I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. Gos. Ped. Inst. Unianowsk **30** (1989), 26-37.
3. S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3**(1922), 133-181.
4. T. Gnana Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA, **65(7)** (2006), 1379-1393.
5. L. Ciric and V. Lakshmikantham, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis: TMA, **70(12)** (2009), 4341-4349.
6. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Mathematica Et Informatica Universitatis Ostraviensis **1** (1993), 5-11.
7. S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti. Semin. Mat. Fis. Univ. Modena **46** (1998), 263-276.
8. D. Guo and V. Lakshmikantham, *Coupled fixed point of nonlinear operator with application*, Nonlinear Anal. TMA., **11** (1987), 623-632.

9. R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures, **7**, No. 1-2, (1999), 71-83.
10. J. K. Kim and S. Chandok, *Coupled common fixed point theorems for generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces*, Fixed Point Theory Appl. (2013), 2013:307.
11. N. V. Luong and N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011), 983-992.
12. N. V. Luong and N. X. Thuan, *Coupled fixed points theorems for mixed monotone mappings and an application to integral equations*, Comput. Math. Appl. **62** (2011), 4238-4248.
13. S. G. Matthews, Partial metric topology, Research report 2012, Dept. Computer Science, University of Warwick, 1992.
14. S. G. Matthews, *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, **728** (1994), 183-197.
15. H. K. Nashine and W. Shatanawi, *Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces*, Comput. Math. Appl. **62** (2011), 1984-1993.
16. H. K. Nashine, J. K. Kim, A. K. Sharma and G. S. Saluja, *Some coupled fixed point without mixed monotone mappings*, Nonlinear Funct. Anal. Appl. **21(2)** (2016), 235-247.
17. J. O. Olaleru, G. A. Okeke and H. Akewe, *Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces*, Int. J. Math. Model. Comput. **2(2)** (2012), 87-98.
18. S. Reich and A. J. Zaslavski, *Well posedness of fixed point problem*, Far East J. Math. special volume part III (2001), 393-401.
19. F. Sabetghadam, H. P. Mashiha and A. H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. (2009), Article ID 125426, 8 pages.
20. B. Samet, *Coupled fixed point theorems for a generalized $Meir - Keeler$ contractions in a partially ordered metric spaces*, Nonlinear Anal. **72(12)** (2010), 4508-4517.
21. B. Samet and H. Yazidi, *Coupled fixed point theorems in a partial order ϵ-chainable metric spaces*, J. Math. Comput. Sci. **1(3)** (2010), 142-151.
22. S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math. (2013). DOI:10.1007/s00009-013-0327-4.
23. O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topl. **6(2)** (2005), 229-240.

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