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SOME RESULTS ON ALMOST GENERALIZED $(\alpha, \beta, \psi, \varphi)$-GERAGHTY TYPE CONTRACTIVE MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper, first, we introduce the notion of almost generalized $(\alpha, \beta, \psi, \varphi)$-Geraghty type contractive mappings and then establish some results for such mappings in the context of partial metric spaces. Our results generalize various known results in this direction. Some examples are presented to illustrate the validity of the results.

1. Introduction and preliminaries

Matthews [2] presented a very interesting generalization of the metric space known as partial metric space (PMS) in which the self distance need not be equal to zero and proved the partial metric version of Banach fixed point theorem. In 1973, Geraghty [1] generalized the Banach contraction principle by considering an auxiliary function. Further, Samet et al. [4] extended and generalized the Banach contraction principle by introducing a new class of contractive type mappings known as $\alpha - \psi$ contractive type mappings. Karapinar and Samet [5] generalized the $\alpha - \psi$ contractive type mappings and established various fixed point theorems. Also Karapinar [6] introduced $\alpha - \psi$-Geraghty contraction type mappings. Thereafter, Rosa and Vetro [8] generalized admissible Geraghty type contraction mappings in partial metric spaces and established some interesting results. Afterward, Chandok [7] introduced the concept of (α, β)-admissible Geraghty type contractive mappings and presented some results. Wang et al. [13] introduced some new types of extended Geraghty contractions, called (α, β)-admissible generalized Geraghty F-contractions, and proved some fixed point results for such contractions in the setting of partial b-metric spaces. Chandok et al. [14] considered rational type expressions for a certain type operators on partial metric spaces. The concept of almost contractions was introduced by Berinde [15, 16] on metric spaces. Aydi et al. [17] considered generalized Berinde-type contractions in the context of partial metric spaces.

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Definition 1.1. [14] Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ such that

(a) ψ is non decreasing and continuous;
(b) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.2. [1] Let $\mathcal{S}$ denotes the class of the functions $\tilde{\beta} : [0,\infty) \to [0,1)$ which satisfy the condition $\tilde{\beta}(t_n) \to 1$ implies $t_n \to 0$

In 1973, Geraghty [1] generalized Banach's contraction principle as follows

Theorem 1.3. [1] *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Assume that there exists $\tilde{\beta} \in \mathcal{S}$ such that for all $x, y \in X$*

$$d(Tx,Ty) \leq \tilde{\beta}(d(x,y))d(x,y)$$

Then T has a unique fixed point in X.

Definition 1.4. [4] Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. T is said to α-admissible if

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$$

for all $x, y \in X$.

In 1992, Matthews [2] presented generalization of metric space as follows:

Definition 1.5. ([2]) Let X be a non-empty set. A function $\rho : X \times X \to [0,\infty)$ is said to be a partial metric on X if the following conditions hold:

(i) $x = y$ if and only if $\rho(x,x) = \rho(y,y) = \rho(x,y)$;
(ii) $\rho(x,x) \leq \rho(x,y)$;
(iii) $\rho(x,y) = \rho(y,x)$;
(iv) $\rho(x,y) \leq \rho(x,z) + \rho(z,y) - \rho(z,z)$. for all $x, y, z \in X$.

The set X equipped with the metric ρ defined above is called a partial metric space and it is denoted by (X,ρ) (in short PMS). Each partial metric ρ on X generates a T_0 topology τ_ρ on X, which has a base of the family of open ρ-balls

$$\{B_\rho(x,\epsilon) : x \in X, \epsilon > 0\}$$

Where

$$B_\rho(x,\epsilon) = \{y \in X : \rho(x,y) < \rho(x,x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$

Example 1.6. ([9]) Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and define $\rho([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,ρ) is a partial metric space.

Example 1.7. ([9]) Let $X = [0,\infty)$ and define $\rho(x,y) = \max\{x,y\}$. Then Then (X,ρ) is a partial metric space.

Lemma 1.8. *([2]) Let (X,ρ) be a partial metric space.*
(a) A sequence $\{x_n\}$ in (X,ρ) converges to a point $x \in X$ if and only if

$$\rho(x,x) = \lim_{n\to\infty} \rho(x_n,x),$$

(b) A sequence $\{x_n\}$ *in* (X, ρ) *is a Cauchy sequence if* $\lim_{m,n\to\infty} \rho(x_n, x_m)$ *exists and finite ,*

(c) (X, ρ) *is complete if every Cauchy* $\{x_n\}$ *in* X *converges to a point* $x \in X$*, such that*

$$\rho(x, x) = \lim_{m,n\to\infty} \rho(x_m, x_n) = \lim_{n\to\infty} \rho(x_n, x) = \rho(x, x).$$

Lemma 1.9. *([10],[2],[3]) Let* ρ *be a partial metric on* X*, then the functions* $d^\rho, d^m : X \times X \to \mathbb{R}^+$ *such that*

$$d^\rho(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y)$$

and

$$\begin{aligned} d^m(x, y) =& \max\{\rho(x, y) - \rho(x, x), \rho(x, y) - \rho(y, y)\} \\ =& \rho(x, y) - \min\{\rho(x, x), \rho(y, y)\} \end{aligned}$$

are metric on X*. Furthermore* (X, d^ρ) *and* (X, d^m) *are metric spaces. It is clear that* d^ρ *and* d^m *are equivalent.*
Let (X, ρ) *be a partial metric space. Then*

(1) A sequence $\{x_n\}$ *in* (X, ρ) *is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space* (X, d^ρ)*,*

(2) (X, ρ) *is complete if and only if the metric space* (X, d^ρ) *is complete. Moreover*
$\lim_{n\to\infty} d^p(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1.10. *([8]) Assume that* $x_n \to z$ *as* $n \to \infty$ *in a partial metric space* (X, ρ) *such that* $\rho(z, z) = 0$ *Then* $\lim_{n\to\infty} \rho(x_n, y) = \rho(z, y)$ *for every* $y \in X$*.*

Lemma 1.11. [8] *If* $\{x_n\}$ *with* $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0$ *is not a Cauchy sequence in* (X, ρ)*, and two sequences* $\{n(k)\}$ *and* $\{m(k)\}$ *of positive integers such that* $n(k) > m(k) > k$*, then following four sequences*

$$\begin{aligned} &\rho(x_{m(k)}, x_{n(k)+1}), \rho(x_{m(k)}, x_{n(k)}), \\ &\rho(x_{m(k)-1}, x_{n(k)+1}), \rho(x_{m(k)-1}, x_{n(k)}) \end{aligned}$$

tend to $\mu > 0$ *when* $k \to \infty$

Lemma 1.12. [14] *Let* (X, ρ) *be a partial metric space.*

(1) if $\rho(x, y) = 0$ *then* $x = y$ *,*

(2) If $x \neq y$ *then* $\rho(x, y) > 0$

Recently, Rosa and Vetro [8] extended Geraghty contraction in partial metric spaces and proved following result:

Theorem 1.13. [8] *Let* (X, ρ) *be a complete partial metric space and* $\alpha : X \times X \to [0, \infty)$ *be a function. Let* $T : X \to X$ *be a self mapping. Suppose that there exists* $\tilde{\beta} \in \mathcal{S}$ *such that*

$$\alpha(x, Tx)\alpha(y, Ty)\rho(M(x, y)) \leq \tilde{\beta}(M(x, y))M(x, y)$$

for all $x, y \in X$ *and*

$$M(x,y) = \{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}$$

Assume that the following conditions hold:

(i) T *is* α*-admissible;*
(ii) There exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(iii) If $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$*, Then either* X *is* α*-regular we have* $\alpha(x_n, x_m) \geq 1$ *with* $n < m$ *or* $x_n \to x$ *we have* $\alpha(x, Tx) \geq 1$ *for all* $m, n \in \mathbb{N}$*;*
(iv) $\alpha(x,y) \geq 1$ *for all* $x, y \in Fix(T)$ *or* $\alpha(x, Tx) \geq 1$ *for all* $x \in Fix(T)$ *where* $Fix(T)$ *is set of fixed points of* T*.*

Then T has a unique fixed point in X.

Definition 1.14. [11] Let Φ be the family of all functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) $\varphi(t) < \psi(t)$ for all $t > 0$
(ii) $\varphi(0) = 0$

Definition 1.15. [12] Let Ω be the family of all functions $\theta : [0,\infty) \to [0,1]$ satisfying the following conditions:

(i) $\theta(t) < 1$ for $t > 0$
(ii) $\lim_{n\to\infty} \theta(t_n) = 1$ implies $\lim_{n\to\infty}(t_n) = 0$

2. Main Results

Definition 2.1. [7] Let T be a self mapping on X and let $\alpha, \beta : X \times X \to [0,\infty)$. We say T is an (α, β) admissible mapping if $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ for all $x, y \in X$.

Definition 2.2. Let (X, ρ) be a complete partial metric space and $T : X \to X$ and $\alpha, \beta : X \times X \to [0,\infty)$ be the mappings.We say that X is (α, β) regular if $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$, $\alpha(x_n, x_{n+1}) \geq 1$, $\beta(x_n, x_{n+1}) \geq 1$ for all n, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, $\beta(x_{n(k)}, x) \geq 1$ for all k.

Definition 2.3. Let (X, ρ) be a partial metric space and $T : X \to X$ be a given self map. We say that T is almost generalized $(\alpha, \beta, \psi, \varphi)$-Geraghty type contractive mapping if there exists $\alpha, \beta : X \times X \to [0,\infty)$ and $\psi \in \Psi$, $\varphi \in \Phi$, $\theta \in \Omega$ and $L \geq 0$ such that for all $x, y \in X$ we have

$$\alpha(x,y)\beta(x,y)\psi(\rho(Tx,Ty)) \leq \theta(\psi(\tilde{\mathcal{M}}(x,y)))\varphi(\tilde{\mathcal{M}}(x,y)) + L\psi(\tilde{\mathcal{N}}(x,y)) \quad (2.1)$$

Where

$$\tilde{\mathcal{M}}(x,y) = \max\left\{\rho(x,y), \rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}, \rho(x,Tx)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}\right\} \quad (2.2)$$

and

$$\tilde{\mathcal{N}}(x,y) = \min\{d^m(x,Tx), d^m(x,Ty), d^m(y,Tx)\} \quad (2.3)$$

Theorem 2.4. *Let (X, ρ) be a complete partial metric space and $T : X \to X$ be self mapping. Suppose $\alpha, \beta : X \times X \to [0, \infty)$ be the mappings satisfying the conditions:*

(i) T is (α, β) admissible;
(ii) T is almost generalized $(\alpha, \beta, \psi, \varphi)$-Geraghty type contractive mapping;
(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(iv) T is continuous;

Then T has a fixed point in X.

Proof. Let x_0 be an arbitrary point such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Suppose we have a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the existence part of the proof is finished. Suppose $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$ Then $\rho(x_n, x_{n+1}) > 0$ by lemma 1.12. Now, since T is (α, β) admissible, so

$$\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$
$$\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1$$

and using induction we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Similarly, we have $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Now, from (2.1) we have

$$\begin{aligned}\psi(\rho(x_n, x_{n+1})) = \psi(\rho(Tx_{n-1}, Tx_n)) &\leq \alpha(x_{n-1}, x_n)\beta(x_{n-1}, x_n)\psi(\rho(Tx_{n-1}, Tx_n)) \\ &\leq \theta(\psi(\tilde{\mathcal{M}}(x_{n-1}, x_n)))\varphi(\tilde{\mathcal{M}}(x_{n-1}, x_n)) + L\psi(\tilde{\mathcal{N}}(x_{n-1}, x_n)) \qquad (2.4)\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{M}}(x_{n-1}, x_n) &= \max\left\{\rho(x_{n-1}, x_n), \rho(x_n, Tx_n)\frac{1+\rho(x_{n-1}, Tx_{n-1})}{1+\rho(x_{n-1}, x_n)}, \rho(x_{n-1}, Tx_{n-1})\frac{1+\rho(x_{n-1}, Tx_{n-1})}{1+\rho(x_{n-1}, x_n)}\right\} \\ &= \max\left\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n+1})\frac{1+\rho(x_{n-1}, x_n)}{1+\rho(x_{n-1}, x_n)}, \rho(x_{n-1}, x_n)\frac{1+\rho(x_{n-1}, x_n)}{1+\rho(x_{n-1}, x_n)}\right\} \\ &= \max\left\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\right\} \qquad (2.5)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(x_{n-1}, x_n) &= \min\{d^m(x_{n-1}, Tx_{n-1}), d^m(x_{n-1}, Tx_n), d^m(x_n, Tx_{n-1})\} \\ &= \min\{d^m(x_{n-1}, x_n), d^m(x_{n-1}, x_{n+1}), d^m(x_n, x_n)\} \\ &= 0 \qquad (2.6)\end{aligned}$$

Now, using (2.5), (2.6) in (2.4) and the fact that and $\psi(t) = 0$ if and only if $t = 0$, we get that

$$\psi(\rho(x_n, x_{n+1})) \leq \theta(\psi(\max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\}))\varphi(\max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n\}) \tag{2.7}$$

Now, if $\rho(x_n, x_{n+1}) > \rho(x_{n-1}, x_n)$ using definitions that $\psi(t) = 0$ if and only if $t = 0$, $\theta(t) < 1$ for $t > 0$ and $\varphi(t) < \psi(t)$ for $t > 0$ we get

$$\begin{aligned}\psi(\rho(x_n, x_{n+1})) &\leq \theta(\psi(\rho(x_n, x_{n+1})))\varphi(\rho(x_n, x_{n+1})) < \theta(\psi(\rho(x_n, x_{n+1})))\psi(\rho(x_n, x_{n+1})) \\ &< \psi(\rho(x_n, x_{n+1}))\end{aligned}$$

which is a contradiction. Therefore, we get that $\{\rho(x_n, x_{n+1}) : n \in \mathbb{N}\}$ is a decreasing sequence of non negative real numbers. Hence it is convergent to a real number, therefore there exists $\lambda_0 \geq 0$ such that

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = \lambda_0$$

Let $\lambda_0 > 0$. Then from (2.7) we get

$$\psi(\rho(x_n, x_{n+1})) \leq \theta(\psi(\rho(x_{n-1}, x_n)))\varphi(\rho(x_{n-1}, x_n)) < \theta(\psi(\rho(x_{n-1}, x_n)))\psi(\rho(x_{n-1}, x_n))$$

Then we have

$$\frac{\psi(\rho(x_n, x_{n+1}))}{\psi(\rho(x_{n-1}, x_n))} < \theta(\psi(\rho(x_{n-1}, x_n))) < 1$$

Now taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(\psi(\rho(x_{n-1}, x_n))) = 1$$

From definitions of θ and ψ we have

$$\lim_{n\to\infty} \psi(\rho(x_{n-1}, x_n)) = 0 \Rightarrow \lim_{n\to\infty} \rho(x_{n-1}, x_n) = 0$$

This is contradiction. Hence

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0 \tag{2.8}$$

Now by lemma 1.9, we have

$$\begin{aligned}d^\rho(x_n, x_{n+1}) =& 2\rho(x_n, x_{n+1}) - \rho(x_n, x_n) - \rho(x_{n+1}, x_{n+1}) \\ =& 2\rho(x_n, x_{n+1}) - \min\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \\ \leq& 2\rho(x_n, x_{n+1})\end{aligned}$$

Therefore by lemma 1.9,

$$\lim_{n\to\infty} d^\rho(x_n, x_{n+1}) = 0 \tag{2.9}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X i.e. We prove that $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.
We prove it by contradiction.
Let

$$\lim_{n\to\infty} \rho(x_n, x_m) \neq 0$$

.

Then sequences in lemma 1.11 tends to $\mu > 0$, when $k \to \infty$.
So we can see that

$$\lim_{k\to\infty} \rho(x_{m(k)}, x_{n(k)}) = \mu \tag{2.10}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is smallest integer with $n(k) > m(k) > k$. Then

$$\lim_{k\to\infty} \rho(x_{n(k)-1}, x_{m(k)}) = \mu \tag{2.11}$$

Again,

$$\rho(x_{m(k)-1}, x_{n(k-1)}) \leq \rho(x_{m(k)-1}, x_{n(k)}) + \rho(x_{n(k)}, x_{n(k)-1}) - \rho(x_{n(k)}, x_{n(k)})$$

Letting $k \to \infty$ and using lemma 1.11 we get

$$\lim_{k\to\infty} \rho(x_{m(k)-1}, x_{n(k-1)}) = \mu \tag{2.12}$$

Again note that

$$\begin{aligned} d^m(x_{n(k)-1}, x_{n(k)}) =& \rho(x_{n(k)-1}, x_{n(k)}) - \min\{\rho(x_{n(k)-1}, x_{n(k)-1}), \rho(x_{n(k)}, x_{n(k)})\} \\ &\leq \rho(x_{n(k)-1}, x_{n(k)}) \end{aligned}$$

So, referring to (2.8)

$$\lim_{k\to\infty} d^m(x_{n(k)-1}, x_{n(k)}) = 0 \tag{2.13}$$

Now in (2.1) replacing x by $x_{n(k)}$ and y by $x_{m(k)}$ respectively, we get

$$\begin{aligned} \psi(\rho(x_{n(k)}, x_{m(k)})) &= \psi(\rho(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \alpha(x_{n(k)-1}, x_{m(k)-1})\beta(x_{n(k)-1}, x_{m(k)-1})\psi(\rho(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \theta(\psi(\tilde{\mathcal{M}}(x_{n(k)-1}, x_{m(k)-1})))\varphi(\tilde{\mathcal{M}}(x_{n(k)-1}, x_{m(k)-1})) + L\psi(\tilde{\mathcal{N}}(x_{n(k)-1}, x_{m(k)-1})) \end{aligned} \tag{2.14}$$

Where

$$\begin{aligned} \tilde{\mathcal{M}}(x_{n(k)-1}, x_{m(k)-1}) =& \max\left\{\rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{m(k)-1}, Tx_{m(k)-1})\frac{1+\rho(x_{n(k)-1}, Tx_{n(k)-1})}{1+\rho(x_{n(k)-1}, x_{m(k)-1})},\right. \\ &\left.\rho(x_{n(k)-1}, Tx_{n(k)-1})\frac{1+\rho(x_{n(k)-1}, Tx_{n(k)-1})}{1+\rho(x_{n(k)-1}, x_{m(k)-1})}\right\} \\ =& \max\left\{\rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{m(k)-1}, x_{m(k)})\frac{1+\rho(x_{n(k)-1}, x_{n(k)})}{1+\rho(x_{n(k)-1}, x_{m(k)-1})},\right. \\ &\left.\rho(x_{n(k)-1}, x_{n(k)})\frac{1+\rho(x_{n(k)-1}, x_{n(k)})}{1+\rho(x_{n(k)-1}, x_{m(k)-1})}\right\} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(x_{n(k)-1}, x_{m(k)-1}) &= \min\{d^m(x_{n(k)-1}, Tx_{n(k)-1}), d^m(x_{n(k)-1}, Tx_{m(k)-1}),\\ &\quad d^m(x_{m(k)-1}, Tx_{n(k)-1})\}\\ &= \min\{d^m(x_{n(k)-1}, x_{n(k)}), d^m(x_{n(k)-1}, x_{m(k)}),\\ &\quad d^m(x_{m(k)-1}, x_{n(k)})\} \qquad (2.16)\end{aligned}$$

Letting $k \to \infty$ in (2.15) and (2.16) and using (2.10),(2.11), (2.12),(2.13) and lemma 1.11 we get

$$\lim_{k\to\infty} \tilde{\mathcal{M}}(x_{n(k)-1}, x_{m(k)-1}) = \mu \qquad (2.17)$$

and

$$\lim_{k\to\infty} \tilde{\mathcal{N}}(x_{n(k)-1}, x_{m(k)-1}) = 0 \qquad (2.18)$$

Now Letting $k \to \infty$ in (2.14) and using (2.17) and (2.18) we get

$$\psi(\mu) \leq \theta(\psi(\mu))\varphi(\mu) < \theta(\psi(\mu))\psi(\mu) < \psi(\mu)$$

This is a contradiction, Therefore

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = 0 \qquad (2.19)$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, ρ) which is complete. Therefore the sequence $\{x_n\}$ is convergent in the space (X, ρ), say $\lim_{n\to\infty} \rho(x_n, z) = 0$. Again from Lemma 1.8, we get

$$\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{m,n\to\infty} \rho(x_n, x_m) = 0$$

Moreover, As T is continuous, we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz$$

□

In the following, we omit the continuity assumption of T in Theorem 2.4.

Theorem 2.5. *Let (X, ρ) be a complete partial metric space and $T : X \to X$ be self mapping. Suppose $\alpha, \beta : X \times X \to [0, \infty)$ be the mappings satisfying the conditions:*

(i) T is (α, β) admissible;
(ii) T is almost generalized $(\alpha, \beta, \psi, \varphi)$-Geraghty type contractive mapping;
(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(iv) X is (α, β) regular;

Then T has a fixed point in X. Further if z, z_1 are fixed points of T with $\alpha(z, Tz) \geq 1$, $\alpha(z_1, Tz_1) \geq 1$ and $\beta(z, Tz) \geq 1$, $\beta(z_1, Tz_1) \geq 1$, then $z = z_1$.

Proof. From the proof of the Theorem 2.4, the sequence x_n defined by $x_{n+1} = Tx_n$ is Cauchy in X. Now, suppose that X is (α, β) regular. We have to show that $Tz = z$. Assume that $\rho(Tz, z) > 0$. Then as X is (α, β) regular there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$, $\beta(x_{n(k)}, z) \geq 1$ for all k. Now in (2.1) replacing x by $x_{n(k)}$ and y by z we get

$$\begin{aligned}\psi(\rho(x_{n(k+1)},Tz)) &= \psi(\rho(Tx_{n(k)},Tz))\\ &\leq \alpha(x_{n(k)},z)\beta(x_{n(k)},z)\psi(\rho(Tx_{n(k)},Tz))\\ &\leq \theta(\psi(\tilde{\mathcal{M}}(x_{n(k)},z)))\varphi(\tilde{\mathcal{M}}(x_{n(k)},z)) + L\psi(\tilde{\mathcal{N}}(x_{n(k)},z)) \quad (2.20)\end{aligned}$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}(x_{n(k)},z) =& \max\left\{\rho(x_{n(k)},z),\rho(z,Tz)\frac{1+\rho(x_{n(k)},Tx_{n(k)})}{1+\rho(x_{n(k)},z)},\right.\\ &\left.\rho(x_{n(k)},Tx_{n(k)})\frac{1+\rho(x_{n(k)},Tx_{n(k)})}{1+\rho(x_{n(k)},z)}\right\}\\ =& \max\left\{\rho(x_{n(k)},z),\rho(z,Tz)\frac{1+\rho(x_{n(k)},x_{n(k)+1})}{1+\rho(x_{n(k)},z)},\right.\\ &\left.\rho(x_{n(k)},x_{n(k)+1})\frac{1+\rho(x_{n(k)},x_{n(k)+1})}{1+\rho(x_{n(k)},z)}\right\} \quad (2.21)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(x_{n(k)},z) &= \min\{d^m(x_{n(k)},Tx_{n(k)}),d^m(x_{n(k)},Tz),d^m(z,Tx_{n(k)})\}\\ &= \min\{d^m(x_{n(k)},x_{n(k)+1}),d^m(x_{n(k)},Tz),d^m(z,x_{n(k)+1})\} \quad (2.22)\end{aligned}$$

Now, taking $k\to\infty$ in (2.21) and ((2.22) we get

$$\lim_{k\to\infty}\tilde{\mathcal{M}}(x_{n(k)},z) = \rho(z,Tz) \quad (2.23)$$

and

$$\lim_{k\to\infty}\tilde{\mathcal{N}}(x_{n(k)},z) = 0 \quad (2.24)$$

Now, taking $k\to\infty$ in (2.20) and using (2.23), (2.24) and definitions of ψ, θ and φ we get

$$\psi(\rho(z,Tz)) \leq \theta(\psi(\rho(z,Tz)))\varphi(\rho(z,Tz)) < \theta(\psi(\rho(z,Tz)))\psi(\rho(z,Tz)) < \psi(\rho(z,Tz))$$

which is a contradiction. Therefore $Tz = z$ i.e. z is a fixed point.
Further, suppose z and z_1 be two fixed point of T such that $\alpha(z,z_1)\geq 1$, $\beta(z,z_1)\geq 1$ and $\rho(z,z_1)>0$ then replacing x by z and y by z_1 in (2.1) we get

$$\begin{aligned}\psi(\rho(z,z_1)) = \psi(\rho(Tz,Tz_1)) &\leq \alpha(z,z_1)\beta(z,z_1)\rho(Tz,Tz_1)\\ &\leq \theta(\psi(\tilde{\mathcal{M}}(z,z_1)))\varphi(\tilde{\mathcal{M}}(z,z_1)) + L\psi(\tilde{\mathcal{N}}(z,z_1))\end{aligned} \quad (2.25)$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}(z,z_1) &= \max\left\{\rho(z,z_1), \rho(z_1,Tz_1)\frac{1+\rho(z,Tz)}{1+\rho(z,z_1)}, \rho(z,Tz)\frac{1+\rho(z,Tz)}{1+\rho(z,z_1)}\right\}\\ &= \max\left\{\rho(z,z_1), \rho(z_1,z_1)\frac{1+\rho(z,z)}{1+\rho(z,z_1)}, \rho(z,z)\frac{1+\rho(z,z)}{1+\rho(z,z_1)}\right\}\\ &= \rho(z,z_1) \qquad (2.26)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(z,z_1) &= \min\{d^m(z,Tz), d^m(z,Tz_1), d^m(z_1,Tz)\}\\ &= \min\{d^m(z,z), d^m(z,z_1), d^m(z_1,z)\}\\ &= 0 \qquad (2.27)\end{aligned}$$

putting (2.26), (2.27) in (2.25) and using the facts that $\psi(t) = 0$ if and only if $t = 0$, $\theta(t) < 1$ for $t > 0$ and $\varphi(t) < \psi(t)$ for $t > 0$ we get

$$\psi(\rho(z,z_1)) \leq \theta(\psi(\rho(z,z_1)))\varphi(\rho(z,z_1)) < \theta(\psi(\rho(z,z_1)))\psi(\rho(z,z_1)) < \psi(\rho(z,z_1))$$

which is a contradiction. Hence T has a unique fixed point. This completes the proof. □

Following are consequences of the theorems.

Corollary 2.6. *Let (X,ρ) be a complete partial metric space. $T : X \to X$ be self-mapping satisfying the condition*

$$\rho(Tx,Ty) \leq \theta(\tilde{\mathcal{M}}(x,y))\tilde{\mathcal{M}}(x,y) + L(\tilde{\mathcal{N}}(x,y)) \qquad (2.28)$$

For all $x, y \in X$, $L \geq 0$, $\theta \in \Omega$ and

$$\tilde{\mathcal{M}}(x,y) = \max\left\{\rho(x,y), \rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}, \rho(x,Tx)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}\right\} \qquad (2.29)$$

and

$$\tilde{\mathcal{N}}(x,y) = \min\{d^m(x,Tx), d^m(x,Ty), d^m(y,Tx)\} \qquad (2.30)$$

Then T has a unique fixed point in X.

Corollary 2.7. *Let (X,ρ) be a complete partial metric space. $T : X \to X$ be self-mapping satisfying the condition*

$$\rho(Tx,Ty) \leq \theta(\rho(x,y))\rho(x,y) \qquad (2.31)$$

For all $x, y \in X$, $\theta \in \Omega$

Then T has a unique fixed point in X.

Example 2.8. Let $X = [0,\infty)$ and $\rho(x,y) = \max\{x,y\}$. Then (X,ρ) is a complete partial metric space. Consider the mapping $T : X \to X$ defined by

$$T(x) = \begin{cases} x - \frac{7}{8} & x \geq 1\\ \frac{x}{8} & 0 \leq x \leq 1\end{cases} \qquad (2.32)$$

and let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be such that $\psi(t) = t$ and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$ and $\theta : [0,\infty) \to [0,1]$ be $\theta(t) = \frac{1}{1+t}$. If we define the functions $\alpha, \beta : X \times X \to [0,\infty)$ as

$$\alpha(x,y) = \begin{cases} \frac{3}{2} & x, y \in [0,1] \\ o & \text{otherwise} \end{cases} \tag{2.33}$$

and

$$\beta(x,y) = \begin{cases} 1 & x, y \in [0,1] \\ \frac{1}{4} & \text{otherwise} \end{cases} \tag{2.34}$$

We show that contractive condition of Theorem 2.4 is satisfied. Without loss of generality we assume that $x \geq y$. Then for $x, y \in [0,1]$ we get

$$\begin{aligned} \alpha(x,y)\beta(x,y)\psi(\rho(Tx,Ty)) &= \alpha(x,y)\beta(x,y)\psi(\rho(\frac{x}{8},\frac{y}{8})) \\ &= \frac{3}{2}(\frac{x}{8}) = \frac{3x}{16} \end{aligned} \tag{2.35}$$

On the other side

$$\begin{aligned} \tilde{\mathcal{M}}(x,y) =& \max\left\{\rho(x,y), \rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}, \rho(x,Tx)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}\right\} \\ =& \max\left\{\rho(x,y), \rho(y,\frac{y}{8})\frac{1+\rho(x,\frac{x}{8})}{1+\rho(x,y)}, \rho(x,\frac{x}{8})\frac{1+\rho(x,\frac{x}{8})}{1+\rho(x,y)}\right\} \\ =& x \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} \tilde{\mathcal{N}}(x,y) &= \min\{d^m(x,Tx), d^m(x,Ty), d^m(y,Tx)\} \\ &= \min\{d^m(x,\frac{x}{8}), d^m(x,\frac{y}{8}), d^m(y,\frac{x}{8})\} \end{aligned} \tag{2.37}$$

Therefore from (2.36) we get

$$\begin{aligned} \theta(\psi(\tilde{\mathcal{M}}(x,y)))\varphi(\tilde{\mathcal{M}}(x,y)) + L\psi(\tilde{\mathcal{N}}(x,y))) &= \theta(\psi(x))\varphi(x) + L\psi(\tilde{\mathcal{N}}(x,y))) \\ &= \theta(x)(\frac{x}{2}) + L(\tilde{\mathcal{N}}(x,y)) \\ &= \frac{x}{2(1+x)} + L(\tilde{\mathcal{N}}(x,y)) \end{aligned} \tag{2.38}$$

Now since $L(\tilde{\mathcal{N}}(x,y)) = L(\min\{d^m(x,\frac{x}{8}), d^m(x,\frac{y}{8}), d^m(y,\frac{x}{8})\}) \geq 0$ for all $x, y \in X$, and From (2.35) and (2.38) we get

$$\frac{3x}{16} \leq \frac{x}{2(1+x)} + L(\tilde{\mathcal{N}}(x,y)) \tag{2.39}$$

it is clear that it satisfies the condition of Theorem 2.4. Hence T has a fixed point, which in this case is 0.

Now, we show that the contractive condition of Corollary 2.7 is satisfied.

Example 2.9. Let $X = [0, 1]$ and $\rho(x, y) = e^{\max\{x,y\}} - 1$. Then (X, ρ) is complete partial metric space. Define $T : X \to X$ by $T(x) = \frac{x}{6}$ and $\theta(t) = \frac{1}{2}$. Let $x, y \in X$ and assume that $x \geq y$, Then

$$\begin{aligned} \rho(Tx, Ty) &= \rho(\frac{x}{6}, \frac{y}{6}) \\ &= e^{\frac{x}{6}} - 1 \end{aligned} \tag{2.40}$$

On the other hand side

$$\begin{aligned} \theta(\rho(x, y))\rho(x, y) &= \frac{1}{2}\rho(x, y) \\ &= \frac{1}{2}(e^x - 1) \end{aligned} \tag{2.41}$$

From (2.40) and (2.41) we get

$$e^{\frac{x}{6}} - 1 \leq \frac{1}{2}(e^x - 1) \tag{2.42}$$

From (2.42) it is clear that it satisfies the condition of Corollary 2.7. Hence it has a fixed point , which in this case is 0.

References

[1] M. Geraghty, *On contractive mappings*, Proc. Am. Math. Soc., **40** (1973), 604-608.

[2] S. G. Matthews, *Partial metric topology*, Research report 212, Department of Computer Science, University of Warwick, (1992).

[3] S. G. Matthews, *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, **728** (1994), 183-197.

[4] B. Samet, C. Vetro and P. Vetro, *Fixed point theorem for $\alpha - \psi$ contractive type mappings*, Nonlinear Anal., **75** (2012), 2154-2165.

[5] E. Karapinar and B. Samet, *Generalized $\alpha - \psi$ -contractive type mappings and related fixed point theorems with applications*, Abstract and Applied Analysis **2012** (2012), Article ID 793486.

[6] E. Karapinar, *$\alpha - \psi$-Geraghty contraction type mappings and some related fixed point results*, Filomat, **28(1)** (2014), 37-48.

[7] S. Chandok, *Some fixed point theorems for (α, β)-admissible Geraghty type contractive mappings and related results*, Math Sci, **9** (2015), 127–135.

[8] V. L. Rosa and P. Vetro, *Fixed points for Geraghty-contractions in partial metric spaces*, Nonlinear Sci. Appl., **7** (2014), 1-10.

[9] M. Kir and H. Kiziltunc, *Generalized fixed point theorems in partial metric spaces*, European Journal of Pure and Applied Mathematics, **4** (2016), 443-451.

[10] D. Ilić, V. Pavlović and V. Rakocević, *Some new extensions of Banach's contraction principle to partial metric space*, Appl. Math. Lett., **24** (2011), 1326-1330.

[11] H. Qawaqneh, M. Noorani, W. Shatanawi and H. Alsamir, *Common fixed point theorems for generalized Geraghty (α, ψ, ϕ)-quasi contraction type mapping in partially ordered metric-like spaces*, Axioms, **7** (2018), 1-17.

[12] K. A. Singh, M. R. Singh and T. C. Singh, *Fixed point theorems for generalized rational $\alpha - \psi$-Geraghty contraction type mappings in metric spaces*, Electronic Journal of Mathematical Analysis and Applications, **9** (2021), 52-58.

[13] M. Wang, N. Saleem, X. Liu, A. H. Ansari and M. Zhou, *Fixed point of (α, β)-admissible generalized Geraghty F-contraction with application*, Symmetry, **14(5)** (2022).

[14] S. Chandok, D. Kumar and M. S. Khan, *Some results in partial metric space using auxiliary functions*, Applied Mathematics E-Notes, **15** (2015), 233-242.

[15] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Analysis Forum, **9(1)** (2004), 43–53.

[16] V. Berinde, *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian Journal of Mathematics, **24(2)** (2008), 10–19.

[17] H. Aydi, S. H. Amor and E. Karapınar, *Berinde-Type Generalized Contractions on Partial Metric Spaces*, Abstract and Applied Analysis **2013** (2013), Article ID 312479.

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