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FINITE VOLUME SCHEME AND RENORMALIZED SOLUTIONS FOR A NONCOERCIVE ELLIPTIC PROBLEM WITH NEUMANN BOUNDARY CONDITIONS AND L^1-DATA

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Abstract. We prove in this paper that the approximate solution, by the finite volumes method, converges to the unique renormalized solution of convective-diffusive elliptic problem with Neumann boundary conditions and L^1-data. In the first part, we recall formulas and give some notations which are useful for the next of the work. We also, give some definitions and properties on Partial Differentials Equations. In the second part we show the bases principle of the main methods of discretization, more precisely, the finite volume method. In the third part, we study a noncoercive elliptic convection-diffusion equation with Neumann boundary conditions and L^1-data. By adapting the strategy developed in the finite volume method, we show that the approximate solution converges to the unique renormalized solution.

1. Introduction

In this work, we consider the discretization by the cell-centered finite volume method of the following convection-diffusion equation with Neumann boundary conditions:

$$\begin{cases} -\Delta u + \operatorname{div}(vu) + bu = f \text{ in } \Omega, \\ \nabla u \cdot \eta - (v \cdot \eta) u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$, with boundary $\partial\Omega$, η is the outer unit normal vector, $v \in (L^p(\Omega))^d$ with $2 < p < +\infty$ if $d = 2$ and $p = d$ if $d \geq 3$, $b \in L^2(\Omega)$ is a positive function and $f \in L^1(\Omega)$ satisfies the compatibility condition $\int_\Omega f \, dx = 0$.

The theory of renormalized solutions has been introduced in [7] for Boltzmann equations and has been adapted in [15, 16] for elliptic problems with L^1-data. It is well known that the renormalized solutions are a convenient framework for parabolic and elliptic equations with L^1-data which provides in general existence,

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stability and uniqueness results.
Elliptic equations with L^1-data and Dirichlet boundary conditions are widely studied in the literature (see [3, 18, 19]).
In the present paper we have to face to a noncoercive character of the operator $u \longmapsto -\Delta u + \operatorname{div}(vu) + bu$, the homogeneous Neumann boundary conditions and the L^1-data. One of the difficulty in the variational and linear case is that the kernel is nontrivial and that we have to impose an additional condition on the solution u to insure uniqueness result, which is in general $\displaystyle\int_\Omega u\,dx = 0$. Concerning the finite volume discretization, several techniques are developed in [10]. The convergence of the cell-centered finite volume scheme for equation (1.1) has been studied in [11] when v = 0 and with measure data. In [5] the authors consider the case $f \in L^2(\Omega)$ and with Neumann boundary conditions: They prove that the solution of this scheme for equation (1.1) converges to the unique weak solution of (1.1) in the sense

$$\begin{cases} u \in H^1(\Omega), \\ \forall \varphi \in H^1(\Omega),\ \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi - \int_\Omega u\, v \cdot \nabla \varphi = \int_\Omega f\, \varphi. \end{cases} \tag{1.2}$$

In [14] the author studied problem (1.1) with Dirichlet homogeneous boundary conditions.
Recall that a renormalized solution of (1.1) is a measurable function u defined from Ω to $\overline{\mathbb{R}}$, such that u is finite a.e. in Ω and

$$\forall k > 0, T_k(u) \in H^1(\Omega), \tag{1.3}$$

$$\lim_{k\to+\infty} \frac{1}{k} \int_\Omega |\nabla T_k(u)|^2 dx = 0, \tag{1.4}$$

$$\begin{cases} \forall h \in C_c^1(\mathbb{R}), \forall \psi \in H^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \nabla u\, h(u)\, \nabla \psi\, dx + \int_\Omega \nabla u\, \psi\, \nabla u\, h'(u)\, dx - \int_\Omega u\, h(u)\, v \cdot \nabla \psi\, dx \\ \quad \displaystyle - \int_\Omega u\, h'(u)\, \psi v \cdot \nabla u\, dx + \int_\Omega b\, u\, h(u)\, \psi\, dx = \int_\Omega \psi\, h(u)\, f\, dx, \end{cases} \tag{1.5}$$

with T_k the truncate function at height k, defined by $T_k(r) = \max\{-k, \min(k, r)\}$ for all $r \in \mathbb{R}$.
Since h has a compact support, each term of (1.5) is well defined.
The existence and the uniqueness of the renormalized solution to (1.1) for L^1-data is proved in [13] by I. Konaté *et al.*
In the present paper by using the tools developed for finite volume schemes, we adapt the strategy used to deal with the existence of a renormalized solution for elliptic equations with L^1-data or with Neumann boundary conditions (see [5, 6, 15, 16]).
The main originality in the present work is that we pass to the limit in a "renormalized discrete version", this is to say that we take a discrete version of $\varphi h(u)$ as test function in the finite volume scheme. The first difficulty is to establish a

discrete version of the estimate on the energy (1.4). Moreover it is worth noting that in (1.5) all the terms are "truncated" while a discrete version of $\varphi h(u)$ in the finite volume scheme leads to some residual terms which are not "truncated". The second difficulty is then to handle these residual terms. The method developed in [14] allows us to deal with nonlinear version of (1.1) in the sense that the solution of the discrete scheme converges to the unique renormalized solution (see Section 6). The third difficulty is the associated boundary conditions being of Neumann type, because contrary to the homogeneous Dirichlet case, the Poincaré's inequality cannot be used. To overcome this difficulty, we use an appropriate discrete Poincaré-Wirtinger inequality involving the median value.
The rest of the paper is organized as follows. In Section 2, we present the finite volume scheme and the properties of the discrete gradient. Section 3 is devoted to prove several estimates, especially the discrete equivalent to (1.5) which is crucial to pass to the limit in the finite volume scheme. In Section 4, we prove the convergence of the cell-centered finite volume scheme via a density argument, and give some a priori estimates in section 5. In the last section, we prove the convergence of the discrete solution to the unique renormalized solution.

2. Assumptions

Let Ω be a connected open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$. We consider the following nonlinear elliptic problem with Neumann boundary conditions:

$$\begin{cases} -\Delta u + \operatorname{div}(vu) + bu = f \text{ in } \Omega, \\ \nabla u \cdot \eta - (v \cdot \eta)u = 0 \text{ on } \partial\Omega, \end{cases}$$

where η is the outer unit normal to $\partial\Omega$. We assume that

$$v \in L^p(\Omega)^d \; with \; 2 < p < +\infty \; if \; d = 2, \, p = d \; if \; d \geq 3, \tag{2.1}$$

$$f \in L^1(\Omega) \tag{2.2}$$

and f satisfies the compatibility condition

$$\int_\Omega f dx = 0. \tag{2.3}$$

As explained in the Introduction, we deal with solutions whose median is equal to zero. If u is a measurable function, we define the median of u (with respect to the Lebesgue measure), denoted by $\operatorname{med}(u)$ as the set of real numbers t such that

$$\operatorname{meas}\{x \in \Omega : u(x) > t\} \leq \frac{\operatorname{meas}(\Omega)}{2}$$

and

$$\operatorname{meas}\{x \in \Omega : u(x) < t\} \leq \frac{\operatorname{meas}(\Omega)}{2}.$$

It is known that $\mathrm{med}(u)$ is non-empty compact interval (see [20]). Let us explicitly observe that if $0 \in \mathrm{med}(u)$ then

$$\mathrm{meas}\{x \in \Omega : u(x) > 0\} \leq \frac{\mathrm{meas}(\Omega)}{2}$$

and

$$\mathrm{meas}\{x \in \Omega : u(x) < 0\} \leq \frac{\mathrm{meas}(\Omega)}{2}.$$

We denote $\underline{\mathrm{med}}(u)$ by

$$\underline{\mathrm{med}}(u) = \inf\left\{t \in \mathbb{R} : \mathrm{meas}\{x \in \Omega : u(x) > t\} \leq \frac{\mathrm{meas}(\Omega)}{2}\right\} \tag{2.4}$$

and $\overline{\mathrm{med}}(u)$ by

$$\overline{\mathrm{med}}(u) = \sup\left\{t \in \mathbb{R} : \mathrm{meas}\{x \in \Omega : u(x) > t\} \geq \frac{\mathrm{meas}(\Omega)}{2}\right\}. \tag{2.5}$$

We observe that if u is an element of $H^1(\Omega)$ (Ω being a connected domain), the median of u is uniquely determined; $\mathrm{med}(u) = \underline{\mathrm{med}}(u) = \overline{\mathrm{med}}(u)$. However it is not the case for the finite volume approximation of (3.11) which is a piecewise-constant function; the median is then the compact interval of $\mathbb{R}$ $[\underline{\mathrm{med}}(u), \overline{\mathrm{med}}(u)]$.

3. **Finite volume scheme**

As in [5], let us define the admissible mesh in the present work.

Definition 3.1. (Admissible mesh). Let Ω be an open subset of $\mathbb{R}^d$. An admissible mesh $\mathcal{M}$ of Ω is given by a finite family $\mathcal{T}$ of related subsets of Ω. Any $K \in \mathcal{T}$ is called control volume. We impose that every $K \in \mathcal{T}$ is opened, the union of the $\overline{\mathcal{T}}$ is $\overline{\Omega}$ and the interface is in some hyperplane, that is for every distinct $K, L \in \mathcal{T}$, $\overline{K} \cap \overline{L}$ is included in an hyperplane. For $K, L \in \mathcal{T}$ two distinct control volumes, we denote by $K/L := \overline{K} \cap \overline{L}$ this interface. Let $K \in \mathcal{T}$, we can write (N for "neighbour"): $N(K) = \{L \in \mathcal{T}; L \notin K, K/L \neq \emptyset\}$, the set of neighbour of K and $\partial K = \displaystyle\bigcup_{L \in N(K)} K/L$, the edge of K, which is a polygonal in dimension d = 2, polyhedral in dimension $d \geq 3$. Finally, we denote by $|K|$ the Lebesgue measure in d-dimensional and $|\partial K|$ (respectively K/L) for the $(d-1)$-dimensional measure of ∂K (resp. of K/L).
Thus, set $\mathcal{E}$ a finite family of disjoint subsets of $\overline{\Omega}$ contained in affine hyperplanes, called the "edges", and set $P = (x_K)_{K \in \mathcal{T}}$ a family of points in Ω such that :

- each $\sigma \in \mathcal{E}$ is a non-empty open subset of ∂K, for some $K \in \mathcal{T}$,
- by denoting $\mathcal{E}(K) = \{\sigma \in \mathcal{E}; \sigma \in \partial K\}$, one has $\partial K = \bigcup_{\sigma \in \mathcal{E}(K)} \overline{\sigma}$ for all $K \in \mathcal{T}$,
- for all $K \neq L$ in $\mathcal{T}$, either the measure of $\overline{K} \cap \overline{L}$ is null or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for all $\sigma \in \mathcal{E}$, that we denote then $\sigma = K/L$,
- for all $K \in \mathcal{T}$, x_k is in the interior of K,
- for all $\sigma = K/L \in \mathcal{E}$, the line $[x_K, x_L]$ intersects and is orthogonal to σ,

- for all $\sigma \in \mathcal{E}$, $\sigma \subset \partial\Omega \cap \partial K$, the line which is orthogonal to σ and going through x_k intercepts σ.

We denote by $|K|$ (resp. $|\sigma|$) the Lebesgue measure of $K \in \mathcal{T}$ (resp. of $\sigma \in \mathcal{E}$). The unit normal to $\sigma \in \mathcal{E}(K)$ outward to K is denoted by $\eta_{K,\sigma}$. $\mathcal{E}_{int}$ (resp. $\mathcal{E}_{ext}$) is defined as the set of interior (resp. the boundary) edges. For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$, we denote $d_{K,\sigma}$ the Euclidean distance between x_k and σ.
For any $\sigma \in \mathcal{E}$, d_σ is defined by $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, if $\sigma = K/L \in \mathcal{E}_{int}$ (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K.\sigma}$, if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K)$. The size of the mesh, denoted by $h_{\mathcal{M}}$, is defined by $h_{\mathcal{M}} = \sup_{K\in\mathcal{T}} diam(K)$.

We will need discrete Sobolev inequalities (see Proposition 3.3), which depend on the constant ζ appearing in the following assumption.

$$\exists\zeta \text{ such that } \forall K \in \mathcal{T},\ \forall\sigma \in \mathcal{E}_K,\ d_{K.\sigma} \geq \zeta d_\sigma. \tag{3.1}$$

The space of piecewise functions associated to an admissible mesh, denoted by $X(\mathcal{M})$, is defined as the set of functions from Ω to $\mathbb{R}$ wich are constant over each control volume of the mesh.
The estimates on finite-volume schemes are usually performed using an adequate $W^{1,p}$-semi-norm associated with the scheme. Let us define the discrete $W^{1,p}$-semi-norm.

Definition 3.2. (Discrete $W^{1,p}$ norm). Let Ω be an open bounded polygonal subset of $\mathbb{R}^d, d \geq 2$, and let $\mathcal{M}$ be an admissible mesh. For $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ and $p \in [1, +\infty[$, the discrete $W^{1,p}$-semi-norm is defied by

$$|u|_{1,p,\mathcal{M}} = \left(\sum_{\sigma\in\mathcal{E}_{int},\, \sigma=K/L} \frac{|\sigma|}{d_\sigma^{p-1}} |u_K - u_L|^p \right)^{\frac{1}{p}}, \quad \forall u \in X(\mathcal{T})$$

and the discrete $W^{1,p}$-norm is defied by

$$\|u\|_{1,p,\mathcal{M}} = \|u\|_{0,p} + |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T})$$

where $\|u\|_{0,p}$ is the L^p norm for piecewise constant functions, $\forall p \in [1, +\infty[$,

$$\|u\|_{0,p} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}} = \left(\sum_{K\in\mathcal{T}} |K||u_K|^p \right)^{\frac{1}{p}}, \forall u \in X(\mathcal{T}).$$

We present now discrete functional an analysis results. We refer the reader to ([5], Lemma 6.1) for a proof of the following discrete Sobolev inequality.

Proposition 3.3. (*Discrete Sobolev inequality*) . *Let Ω be a bounded polygonal open subset of $\mathbb{R}^d$ and let $\mathcal{M}$ be an admissible mesh satisfying* (3.1). *Let $q < +\infty$ if $d = 2$ and $q = \frac{2d}{d-2}$ if $d \geq 3$. Then there exists $C = C(\Omega, \zeta, q)$ such that, for all $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$,*

$$\|u\|_{0,p} \leq C\big(|u|_{1,2,\mathcal{M}} + \|u\|_{0,2}\big). \tag{3.2}$$

In the already cited references discrete Poincaré and Poincaré-Wirtinger inequalities are related to the discrete space $W_0^{1,p}(\Omega)$ and zero boundary condition or the discrete space $W^{1,p}(\Omega)$ with discrete mean value. We derive here a discrete

Poincaré-Wirtinger inequality involving the median. The proof is given in the appendix.

Proposition 3.4. (*Discrete Poincaré-Wirtinger median inequality*). *Let Ω be an open bounded connected polyhedral domain of $\mathbb{R}^d$ and let $\mathcal{M}$ be an admissible mesh satisfying* (3.1). *Then for $1 \le p < +\infty$ there exists a constant $C > 0$ only depending on Ω, d and p such that*

$$\|u - c\|_{0,p} \le \frac{C}{\zeta^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T}) \tag{3.3}$$

where c belongs to med(u).

Proposition 3.5. *(Discrete Rellich's theorem). Let $(\mathcal{M}_m)_{m\ge1}$ be a sequence of admissible meshes satisfying* (3.1) *and such that $h_{\mathcal{M}_m} \to 0$ as $m \to +\infty$. If $v_m \in X(\mathcal{T}_m)$ is such that $(|v_m|_{1,2,\mathcal{M}} + \|v_m\|_{0,2})$ is bounded, then $(v_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$. Furthermore, any limit in $L^2(\Omega)$ of a subsequence of $(v_m)_{m\in\mathbb{N}}$ belongs to $H^1(\Omega)$.*

Before writing the finite volume scheme, let us define a discrete finite volume gradient (see [14])).

Definition 3.6. (Discrete finite volume gradient) For all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, we define the volume $D_{K,\sigma}$ as the cone of basis σ and of opposite vertex x_K. Then, we define the "diamond-cell" D_σ by:

$$\begin{aligned} D_\sigma &= \quad D_{K,\sigma} \cup D_{L,\sigma} && \text{if } \sigma = K/L \in \mathcal{E}_{int}, \\ D_\sigma &= \qquad D_{K,\sigma} && \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K), \end{aligned}$$

and

$$|D_\sigma| = \frac{|\sigma|\, d_\sigma}{d}.$$

The approximate gradient $\nabla_{\mathcal{M}} v$ of a function $v \in X(\mathcal{T})$ is defied as a piece-wise constant function over each diamond cell and given by

$$\begin{aligned} &\forall \sigma \in \mathcal{E}_{int},\ \sigma = K/L, \quad \nabla_{\mathcal{M}} v(x) = |\sigma| \frac{v_L - v_K}{|D_\sigma|} \eta_{K,\sigma} = d \frac{v_L - v_K}{d_\sigma} \eta_{K,\sigma}, \quad \forall x \in D_\sigma, \\ &\forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K), \qquad\qquad \nabla_{\mathcal{M}} v(x) = 0, \qquad\qquad \forall x \in D_\sigma. \end{aligned}$$

FIGURE 1. Example of control volume for the method of finite volume in two dimensions of space

Let us then give convergence property of the discrete gradient.

Lemma 3.7. *(Weak convergence of the finite volume gradient). Let $(\mathcal{M}_m)_{m\ge1}$ be a sequence of admissible meshes satisfying* (3.1) *and such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. Let $v_m \in X(\mathcal{T}_m)$ and let us assume that there exists $\alpha \in [1, +\infty[$ and $C > 0$ such that $\|v_m\|_{1,\alpha,\mathcal{M}_m} \le C$, and that v_m converges in $L^1(\Omega)$ to $v \in W^{1,\alpha}(\Omega)$. Then $\nabla \mathcal{M}_m v_m$ converges to ∇v weakly in $L^\alpha(\Omega)^d$.*

The principle of finite-volume schemes for convection-diffusion problems is to write a flux balance using quantities $\mathcal{F}_{K,\sigma}$ that approximate $\displaystyle\int_\sigma \big(\nabla u\cdot\eta-(v\cdot\eta)u\big)$. In order to do so we will need discretizations of the fluxes of v through the edges of the mesh:

$$v_{K,\sigma}=\frac{1}{|D_\sigma|}\int_{D_\sigma} v.\eta_{K,\sigma}dx. \tag{3.4}$$

We now need to explain how to construct the approximation of the fluxes $\mathcal{F}_{K,\sigma}$ using only approximate values $(u_K)_{K\in\mathcal{T}}$ of the solution inside the control volumes (these will be the unknowns of the system describing the scheme). In this paper we use the upwind fluxes defined by:

$$\mathcal{F}_{K,\sigma}=\frac{|\sigma|}{d_\sigma}(u_K-u_L)+|\sigma|(v^+_{K,\sigma}\,u_K-v^-_{K,\sigma}\,u_L)\quad\forall\sigma=K/L\in\mathcal{E}_{int}, \tag{3.5}$$

where $s^+=\max(s,0)$ and $s^-=\max(-s,0)$ are the positive and negative parts of a real number s. Assuming that u_K and u_L are approximations of u at x_K and x_L, respectively, the orthogonality between $(x_K,\,x_L)$ and σ ensures that $\dfrac{|\sigma|}{d_\sigma}(u_K-u_L)$ is a consistent approximation of $\displaystyle\int_\sigma -\nabla u\cdot\eta_{K,\sigma}$. The quantity $|\sigma|(v^+_{K,\sigma}\,u_K-v^-_{K,\sigma}\,u_L)$ is a upwind discretization approximating the convective flux $\displaystyle\int_\sigma u\,v\cdot\eta_{K,\sigma}$, which stabilizes the scheme (at the cost of the introduction of an additional numerical diffusion).

Defining $B(s)=1+(-s)^+=1+s^-$, the fluxes of the upwind scheme can be written as

$$\mathcal{F}_{K,\sigma}=\frac{|\sigma|}{d_\sigma}\Big(B(-v_{K,\sigma}\,d_\sigma)\,u_K-B(v_{K,\sigma}\,d_\sigma)\,u_L\Big).$$

As in [5], we note that the function B satisfies the following:

$$B \text{ is Lipschitz-continuous on } \mathbb{R}, \tag{3.6}$$

$$B(0)=1 \text{ and } B(s)>0\quad\forall\,s\in\mathbb{R}, \tag{3.7}$$

$$B(s)-B(-s)=-s\quad\forall\,s\in\mathbb{R}. \tag{3.8}$$

Let $\mathcal{M}$ be an admissible mesh in the sense of Definition 3.1, we can now define the finite volume discretization of (1.1). For $K\in\mathcal{T}$ and $\sigma\in\mathcal{E}(K)$, we define

$$b_K=\frac{1}{|K|}\int_K b\,dx, \tag{3.9}$$

and

$$f_K=\frac{1}{|K|}\int_K f\,dx. \tag{3.10}$$

So, we can write the scheme on (1.1) as the following:
For all $K\in\mathcal{T}$,

$$\sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{int}} \mathcal{F}_{K,\sigma} + |K| b_K u_K = |K| f_K, \tag{3.11}$$

$$\mathcal{F}_{K,\sigma} = \frac{|\sigma|}{d_\sigma}\Big(B(-v_{K,\sigma}\, d_\sigma)\, u_K - B(v_{K,\sigma}\, d_\sigma)\, u_L\Big) \quad \forall\, \sigma = K/L, \tag{3.12}$$

where B is a function satisfying (3.6)-(3.8). The first equation (3.11) comes from the physical balance of fluxes. The second equation (3.12) defines the approximate fluxes. Note that the homogeneous Neumann boundary condition in (1.1) is taken into account in the fact that the sum in (3.11) is restricted to the interior edges of each control volume. In fact, the boundary condition consists of imposing $\mathcal{F}_{K,\sigma} = 0$ for $\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{ext}$.
It will be useful to note that, since $v_{K,\sigma} = -v_{L,\sigma}$ whenever $\sigma = K/L$, the fluxes defined by (3.12) are conservative:

$$\mathcal{F}_{K,\sigma} = -\mathcal{F}_{L,\sigma} \qquad \forall\, \sigma = K/L.$$

Now we are led to give our main results.

Theorem 3.8. *(Existence of a solution for the scheme on (1.1)) Let $\mathcal{M}$ be an admissible mesh in the sense of Definition 3.1 satisfying (3.1). Then, there exists a unique solution $u_\mathcal{M} = (u_K)_{K\in\mathcal{T}}$ to (3.6)-(3.8) and (3.11)-(3.12) having $\underline{med}(u_\mathcal{M}) = 0$.*

Theorem 3.9. *(Convergence of the solution for the scheme on (1.1)) Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes in the sense of Definition 3.1, which satisfy (3.1) and such that $h_{\mathcal{M}_m}$ goes to 0 as $m \to +\infty$. Let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution of (3.6)-(3.8) and (3.11)-(3.12) such that $\underline{med}(u_m) = 0$. Then u_m converges to the unique renormalized solution u of (1.1) having med$(u) = 0$, in the sense that*

$$\begin{gathered} u_m \text{ converges to } u \text{ a.e in } \Omega, \\ \forall n \in \mathbb{N}, \nabla_{\mathcal{M}_m} T_n(u_m) \text{ converges to } \nabla T_n(u), \text{ weakly in } (L^2(\Omega))^d, \end{gathered}$$

as $m \to \infty$.

4. Existence and uniqueness of the solution to the schemes

We intend here to prove Theorem 3.8 by means of linear algebra tools. The scheme (3.6)-(3.8) and (3.11)-(3.12) leads to a linear system of equations that can be written as

$$\mathcal{A}_b U = F, \tag{4.1}$$

where $\mathcal{A}_b = \mathbb{A} + \mathcal{B}\mathbb{D}$, $U = (u_K)_{K\in\mathcal{T}}$, $\mathcal{B} = (|K| b_K)_{K\in\mathcal{T}}$, $F = (|K| f_K)_{K\in\mathcal{T}}$, $\mathbb{D}$ is the diagonal matrix whose diagonal entries are $\mathbb{D}_{K,K} = |K|$ and $\mathbb{A}$ is the square matrix of size $Card(\mathcal{T}) \times Card(\mathcal{T})$ with entries

$$\mathbb{A}_{K,K} = \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma} B(-v_{K,\sigma}\, d_\sigma) \quad \forall K \in \mathcal{T}, \tag{4.2}$$

$$\mathbb{A}_{K,L} = -\frac{|\sigma|}{d_\sigma} B(v_{K,\sigma}\, d_\sigma) \quad \forall K \in \mathcal{T}, \forall L \in N(K), \quad \text{with } \sigma = K/L, \tag{4.3}$$

$$\mathbb{A}_{K,L} = 0 \quad \forall K \in \mathcal{T}, \forall L \notin N(K). \tag{4.4}$$

Theorem (3.8) is thus easy consequence of the following proposition.

Proposition 4.1. *For all $b \in L^2(\Omega)$ a nonnegative function, the diagonal coefficients of $\mathcal{A}_b$ are positive, the extra-diagonal coefficients of $\mathcal{A}_b$ are nonpositive and the sum of the coefficients in each column of $\mathcal{A}_b$ is positive. Therefore $\mathcal{A}_b$ is an M-matrix and is invertible.*

Proof. Due to (4.2)-(4.4) and (3.8) we note that all of the diagonal entries of $\mathbb{A}$ are strictly positive, whereas the extra-diagonal coefficients are nonpositive. This is therefore also the case for $\mathcal{A}_b$.
Moreover, since $v_{L,\sigma} = -v_{K,\sigma}$ whenever $\sigma = K/L \in \mathcal{E}_{int}$, we have

$$\mathbb{A}_{K,K} = -\sum_{L \in N(K)} \mathbb{A}_{L,K} \quad \forall K \in \mathcal{T}. \tag{4.5}$$

In other words, in each column the diagonal term is the opposite of the sum of the extra-diagonal terms. This has the following consequence:
the sum of the coefficients in the column K of $\mathcal{A}_b$ is equal to $\mathcal{B}|K|$, and the proposition is proved. ☐

5. Estimations

In this section, we first establish in Proposition 5.1 an estimate on $\ln(1+|u_{\mathcal{M}}|)$ which is crucial to control the measure of the set $\{|u_{\mathcal{M}}| > n\}$. Then, we show in Proposition 5.3 an estimate on $T_n(u_{\mathcal{M}})$ and the convergence of $T_n(u_{\mathcal{M}})$ to $T_n(u)$. Finally, we prove in Proposition 5.4 a discrete version of the decay of the energy.

Proposition 5.1. *(Estimate on $\ln(1+|u_{\mathcal{M}}|)$ with $\underline{med}\,(u_{\mathcal{M}}) = 0$) Let $\mathcal{M}$ be an admissible mesh satisfying (3.1). If $u_{\mathcal{M}} = (u_K)_{K \in \mathcal{T}}$ is a solution to (3.6)-(3.8) and (3.11)-(3.12), such that $\underline{med}\,(u_{\mathcal{M}}) = 0$, then*

$$\| \ln(1+|u_{\mathcal{M}}|)\|^2_{1,2,\mathcal{M}} \leq C\big(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}} \|v\|^2_{L^p(\Omega)^d}\big), \tag{5.1}$$

where $C = C(\Omega, p, \zeta)$ is a positive constant.

Proof. A log-estimate was obtained in (Proposition 3.1, [14]) in the case of Dirichlet boundary conditions. Since we deal with Neumann boundary conditions and the specific choice of $\underline{med}\,(u_{\mathcal{M}}) = 0$, we will adapt the proof derived in [14]. Let $\varphi(s) = \displaystyle\int_0^s \frac{dt}{(1+|t|)^2}$. Taking $\varphi(u_K)$ as a test function in the scheme (3.11) and reordering the sums yield

$$\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}\big[B(-v_{K,\sigma}\,d_\sigma)u_K - B(v_{K,\sigma}\,d_\sigma)u_L\big](\varphi(u_K)-\varphi(u_L))$$
$$+\sum_{K\in\mathcal{T}}|K|b_K u_K\varphi(u_K) = \sum_{K\in\mathcal{T}}\int_K f\,\varphi(u_K)dx. \tag{5.2}$$

b being nonnegative and $\varphi(s)$ having the same sign as s,

$$\sum_{K\in\mathcal{T}}|K|b_K u_K\varphi(u_K)\geq 0. \tag{5.3}$$

Moreover, since φ is bounded by 1, we deduce that

$$\left|\sum_{K\in\mathcal{T}}\int_K f\ \varphi(u_K)dx\right| \leqslant \|f\|_{L^1(\Omega)}. \tag{5.4}$$

Using (3.5), the first term of (5.2) can be rewritten as

$$\begin{aligned}
&\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}\big[B(-v_{K,\sigma}\,d_\sigma)u_K - B(v_{K,\sigma}\,d_\sigma)u_L\big](\varphi(u_K)-\varphi(u_L))\\
&\quad= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(u_K-u_L)(\varphi(u_K)-\varphi(u_L))\\
&\qquad + \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma|(v^+_{K,\sigma}\,u_K - v^-_{K,\sigma}\,u_L)\big(\varphi(u_K)-\varphi(u_L)\big)\\
&\quad= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(u_K-u_L)(\varphi(u_K)-\varphi(u_L))\\
&\qquad + \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma|\,v_{K,\sigma}\,u_{\sigma,+}(\varphi(u_{\sigma,+})-\varphi(u_{\sigma,-})),
\end{aligned} \tag{5.5}$$

where

$$\forall\sigma=K/L\in\mathcal{E}_{int}, u_{\sigma,+}=u_K \text{ if } v_{K,\sigma}\geq 0 \text{ and } u_{\sigma,+}=u_L,\ \text{otherwise}. \tag{5.6}$$

We denote by $u_{\sigma,-}$ the downstream choice of u which is such that $\{u_{\sigma,+},u_{\sigma,-}\}=\{u_K,u_L\}$ (with $u_L=0$ if $\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}(K)$).
Using (5.3)-(5.5) in (5.2), we get

$$\begin{aligned}
&\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(u_K-u_L)(\varphi(u_K)-\varphi(u_L))\\
&\qquad\leq \|f\|_{L^1(\Omega)} + \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma||v_{K,\sigma}|u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})).
\end{aligned} \tag{5.7}$$

To control the second term in the right-hand side of (5.7) we introduce the set of edges A (see [8]) by

$$A=\{\sigma\in\mathcal{E}_{int};\ u_{\sigma,+}\geq u_{\sigma,-},\ u_{\sigma,+}<0\}\cup\{\sigma\in\mathcal{E}_{int};\ u_{\sigma,+}<u_{\sigma,-},\ u_{\sigma,+}\geq 0\}, \tag{5.8}$$

Since φ is non-decreasing, as in [8] we obtain

$$\sum_{\sigma\in\mathcal{E}_{int}} |\sigma||v_{K,\sigma}|u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})) \leq \sum_{\sigma\in A} |\sigma||v_{K,\sigma}|u_{\sigma,+}(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})). \quad (5.9)$$

Now using Cauchy-Schwarz and Hölder inequalities, and the following inequality (see Lemma 3.1, [9]),

$$\forall\sigma\in A,\ |u_{\sigma,+}|^2|\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|^2 \leq |u_{\sigma,-}-u_{\sigma,+}||\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|,$$

we obtain

$$\begin{aligned}
\sum_{\sigma\in A}|\sigma||v_{K,\sigma}||u_{\sigma,+}|(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})) &\leq \left(\sum_{\sigma\in A}|\sigma|d_\sigma|v_{K,\sigma}|^2\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{\sigma\in A}\frac{|\sigma|}{d_\sigma}|u_{\sigma,+}|^2|\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\sigma\in A}|\sigma|d_\sigma\right)^{\frac{p-2}{p}}\left(\sum_{\sigma\in A}|\sigma|d_\sigma|v_{K,\sigma}|^p\right)^{\frac{1}{p}} \\
&\quad\times\left(\sum_{\sigma\in A}\frac{|\sigma|}{d_\sigma}|u_{\sigma,+}|^2|\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\sigma\in A}|\sigma|d_\sigma\right)^{\frac{p-2}{p}}\left(\sum_{\sigma\in A}|\sigma|d_\sigma|v_{K,\sigma}|^p\right)^{\frac{1}{p}} \\
&\times\left(\sum_{\sigma\in A}\frac{|\sigma|}{d_\sigma}|u_k-u_L||\varphi(u_K)-\varphi(u_L)|\right)^{\frac{1}{2}}. \quad (5.10)
\end{aligned}$$

Recalling that $\sum_{\sigma\in A}|\sigma|d_\sigma \leq \sum_{\sigma\in\mathcal{E}_{int}}|\sigma|d_\sigma = d|\Omega|$ and since the term $\left(\sum_{\sigma\in A}|\sigma|d_\sigma|v_{K,\sigma}|^p\right)^{\frac{1}{p}}$ is bounded by $d^{\frac{1}{p}}\|v\|_{(L^p(\Omega))^d}$, by Young's inequality we get

$$\begin{aligned}
\sum_{\sigma\in A}|\sigma||v_{K,\sigma}||u_{\sigma,+}|(\varphi(u_{\sigma,-})-\varphi(u_{\sigma,+})) &\leq \frac{1}{2\alpha}d|\Omega|^{\frac{p-2}{p}}\|v\|^2_{(L^p(\Omega))^d} \\
&+\frac{\alpha}{2}|\Omega|^{\frac{p-2}{p}}\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(u_K-u_L)(\varphi(u_K)-\varphi(u_L)), \quad (5.11)
\end{aligned}$$

where $\alpha>0$.
Combining (5.7) and (5.11), and choosing α such that $0<\alpha<2|\Omega|^{(2-p)/p}$, gives

$$\begin{aligned}
&\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(u_K-u_L)(\varphi(u_{u_K})-\varphi(u_L)) \\
&\qquad\leq C(\Omega,\alpha,p)\left(2\|f\|_{L^1(\Omega)}+d|\Omega|^{\frac{p-2}{p}}\|v\|^2_{(L^p(\Omega))^d}\right). \quad (5.12)
\end{aligned}$$

Recaling that, for all $(x,y) \in \mathbb{R}^2$, $(\ln(1+|x|)-\ln(1+|y|))^2 = (x-y)(\varphi(x)-\varphi(y))$. Then,

$$\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(\ln(1+|u_K|)-\ln(1+|u_L|))^2$$
$$\leq C(\Omega,\alpha,p)\bigg(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}}\|v\|^2_{(L^p(\Omega))^d}\bigg). \tag{5.13}$$

Since $\underline{\text{med}}(\ln(1+u_{\mathcal{M}})) = 0$, we can use the discrete Poincaré-Wirtinger inequality (3.3) to obtain

$$\|\ln(1+|u_{\mathcal{M}}|)\|^2_{1,2,\mathcal{M}} \leq C\bigg(2\|f\|_{L^1(\Omega)} + d|\Omega|^{\frac{p-2}{p}}\|v\|^2_{(L^p(\Omega))^d}\bigg).$$

□

Let us state a corollary, which is useful in the next.

Corollary 5.2. *(see [8]) Let $\mathcal{M}$ be an admissible mesh. If $u_{\mathcal{M}} = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ is a solution to (3.6)-(3.8) and (3.11)-(3.12) and, for $n>0$, $E_n = \{|u_{\mathcal{M}}| > n\}$, then there exists $C>0$ only depending on (Ω, v, f, d, p) such that*

$$|E_n| \leq \frac{C}{(\ln(1+n))^2}. \tag{5.14}$$

The following proposition checks that our schemes satisfy properties that are well known for the continuous equations, namely, if b is positive then it is easy to obtain a priori estimates for the solution to (1.1) and the L^2-norm of the gradient of the solution to (1.1) is always controlled by the L^2-norm of the solution. Of course, the main difference with respect to the continuous case is that, at the discrete level, we have to make sure that these estimates do not depend on the size of the mesh.

Proposition 5.3. *(Estimation on $T_n(u_{\mathcal{M}})$). Let $\mathcal{M}$ be an admissible mesh verifying (3.1). If $u_{\mathcal{M}} = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ is a solution to (3.6)-(3.8) and (3.11)-(3.12), such that $\underline{med}(u_{\mathcal{M}}) = 0$, then for any $n \geq 0$, there exists $C>0$ only depending on $(\Omega, v, f, n, d, \zeta)$ such that*

$$\|T_n(u_{\mathcal{M}})\|_{1,2,\mathcal{M}} \leq C. \tag{5.15}$$

Moreover, let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes such that there exists $\zeta > 0$ satisfying for all $m \geq 1$, for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, $d_{K\sigma} \geqslant \zeta d_\sigma$, and such that $h_{\mathcal{M}_m}$ goes to zero as $m \to \infty$, and let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution to (3.6)-(3.8) and (3.11)-(3.12), such that $\underline{med}(u_{\mathcal{M}}) = 0$. Then there exists a measurable function u finite a.e. in Ω such that, up to a subsequence, still indexed by m,

$$T_n(u) \in H^1(\Omega), \text{ for any } n > 0, \tag{5.16}$$
$$med(u) = 0, \tag{5.17}$$
$$T_n(u_m) \to T_n(u) \text{ strongly in } L^2(\Omega) \text{ and a.e}, \tag{5.18}$$
$$\nabla_{\mathcal{M}_m} T_n(u_m) \rightharpoonup \nabla T_n(u) \text{ in } (L^2(\Omega))^d, \forall n > 0. \tag{5.19}$$

Proof. The proof is divided into 2 steps. In the first step, we prove that $T_n(u_{\mathcal{M}})$ satisfies the a priori estimate (5.15). In the second step, considering a sequence of admissible meshes $\mathcal{M}_m$, we prove that the solution u_m to the scheme (3.6)-(3.8) and (3.11)-(3.12) converges to a function u as m goes to infinity, that proves (5.16)-(5.19).

Step 1: Estimation on $T_n(u_{\mathcal{M}})$.
We multiply equation (3.11) by $T_n(u_K)$ and sum over $K \in \mathcal{T}$. Due to the conservativity of the fluxes and to (3.12), gathering by edges, we find that

$$\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}\big[B(-v_{K,\sigma}\, d_\sigma)u_K - B(v_{K,\sigma}\, d_\sigma)u_L\big](T_n(u_K) - T_n(u_L))$$
$$+ \sum_{K\in\mathcal{T}} |K| b_K u_K T_n(u_K) = \sum_{K\in\mathcal{T}} \int_K f T_n(u_K) dx. \tag{5.20}$$

Since b is nonnegative and since $rT_n(r) \geqslant 0 \quad \forall r$, we notice that the second term in the left hand side of (5.20) is nonnegative. Moreover, since T_n is bounded by n, we deduce that

$$\left|\sum_{K\in\mathcal{T}} \int_K f T_n(u_K) dx\right| \leqslant n\|f\|_{L^1(\Omega)}.$$

From (5.5), the first term of (5.20) can be rewritten as

$$\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}\big[B(-v_{K,\sigma}\, d_\sigma)u_K - B(v_{K,\sigma}\, d_\sigma)u_L\big](T_n(u_K) - T_n(u_L)) := I_1 + I_2,$$

with

$$\begin{aligned} I_1 &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(u_K - u_L)(T_n(u_K) - T_n(u_L)), \\ I_2 &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma|\, v_{K,\sigma}\, u_{\sigma,+}\big(T_n(u_K) - T_n(u_L)\big), \end{aligned}$$

Therefore, we deduce from (5.20)

$$I_1 \leqslant n\|f\|_{L^1(\Omega)} - I_2.$$

As in the proof of Proposition 5.1, we need the subset A of edges defined in (5.8). Since T_n is non decreasing we have

$$-I_2 = \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma||v_{K,\sigma}|u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})).$$

Notice that $\forall \sigma \in A, |u_{\sigma,+}| \geq n$ implies $|u_{\sigma,-}| \geq n$. So, we deduce that

$$\forall \sigma \in A, u_{\sigma,+}(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})) = T_n(u_{\sigma,+})(T_n(u_{\sigma,-}) - T_n(u_{\sigma,+})).$$

Using Cauchy-Schwarz and Young inequalities, it follows that

$$\begin{aligned}
-I_2 &\leq \sum_{\sigma\in A}|\sigma||v_{K,\sigma}|u_{\sigma,+}T_n(u_{\sigma,+})(T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))\\
&\leq \Big(\sum_{\sigma=K/L\in\mathcal{E}_{int}} d_\sigma|\sigma||v_{K,\sigma}|^2\Big)^{\frac12}\Big(\sum_{\sigma\in A}\frac{|\sigma|}{d_\sigma}T_n(u_{\sigma,+})^2(T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))^2\Big)^{\frac12}\\
&\leq nd^{\frac12}\|v\|_{L^2(\Omega)^d}\Big(\sum_{\sigma\in A}\frac{|\sigma|}{d_\sigma}(T_n(u_{\sigma,-})-T_n(u_{\sigma,+}))^2\Big)^{\frac12}\\
&\leq \frac12 n^2d\|v\|^2_{L^2(\Omega)^d}+\frac12\sum_{\sigma\in A}\frac{|\sigma|}{d_\sigma}\big(T_n(u_{\sigma,-})-T_n(u_{\sigma,+})\big)^2
\end{aligned}$$

Since $T_n(u_K)-T_n(u_L)\leq u_K-u_L$ (because T_n is 1-Lipschitz function), we have

$$-I_2\leq\frac12 n^2d\|v\|^2_{L^2(\Omega)^d}+\frac12\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(u_K-u_L)(T_n(u_K)-T_n(u_L)),$$

and we can deduce that

$$\frac12\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(u_K-u_L)(T_n(u_K)-T_n(u_L))\leq n\|f\|_{L^1(\Omega)}+\frac12 n^2d\|v\|^2_{L^2(\Omega)^d}.$$

Therefore, using again the fact that T_n is 1-Lipschitz, we can write :

$$\frac12\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}\big(T_n(u_K)-T_n(u_L)\big)^2\leq n\|f\|_{L^1(\Omega)}+\frac12 n^2d\|v\|^2_{L^2(\Omega)^d},$$

that is

$$|T_n(u)|^2_{1,2,\mathcal{M}}\leq 2n\|f\|_{L^1(\Omega)}+n^2d\|v\|^2_{L^2(\Omega)^d}. \tag{5.21}$$

Using Discrete Poincaré-Wirtinger median inequality and the fact that $\underline{\text{med}}\,(u_{\mathcal{M}})=0$, we obtain

$$\|T_n(u)\|_{1,2,\mathcal{M}}\leq(1+\frac{C}{\sqrt{\zeta}})|T_n(u)|_{1,2,\mathcal{M}}. \tag{5.22}$$

Combining (5.21) and (5.22), we deduce the desired result (5.15).

step 2: **Basic convergence**

Let $u_m=(u_K^m)_{K\in\mathcal{T}_m}\in X(\mathcal{T}_m)$ be a solution to (3.6)-(3.8) and (3.11)-(3.12), such that $\underline{\text{med}}\,(u_{\mathcal{M}})=0$. According to (5.15), we deduce that the sequence $\Big(\|T_n(u_m)\|_{1,2,\mathcal{M}}\Big)_{m\geq1}$ is uniformly bounded with respect to m. Applying Discrete Rellich's theorem and the diagonal process, up to a subsequence still indexed by m, we deduce that for any $n\in\mathbb{N}$, there exists $v_n\in H^1(\Omega)$ such that

$$T_n(u_m)\longrightarrow v_n,\ \text{a.e. in }\Omega,\ \text{as } m\to\infty. \tag{5.23}$$

We intend to show that u_m converges a.e. to u, by proving that u_m is a Cauchy sequence in measure. Let $\lambda>0$, then for all $n>0$ and all $r,s\geq0$, we have

$$\{|u_r-u_s|>\lambda\}\subset\{|u_r|>n\}\cup\{|u_s|>n\}\cup\{|T_n(u_r)-T_n(u_s)|>\lambda\}.$$

Let $\varepsilon > 0$ fixed. By (5.14), let $n > 0$ such that, for all $r, s \geq 0$,

$$\text{meas}(\{|u_r| > n\}) + \text{ meas}(\{|u_s| > n\}) < \frac{\varepsilon}{2}.$$

Once n is chosen, we deduce from Step 1 that $\Big(T_n(u_m)\Big)_{m\geq 1}$ is a Cauchy sequence in measure, thus

$$\exists M_0 > 0;\, \forall r, s \geq M_0, \quad \text{meas}(\{|T_n(u_r) - T_n(u_s)| > \lambda\}) < \frac{\varepsilon}{2}.$$

Therefore, we deduce that

$$\forall r, s \geq M_0, \quad \text{meas}(\{|u_r - u_s| > \lambda\}) < \varepsilon.$$

Hence $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in measure. Consequently, up to a subsequence still indexed by m, there exists a measurable function u such that

$$u_m \longrightarrow u \text{ a.e. in } \Omega. \tag{5.24}$$

Due to Corollary 5.2, u is finite a.e. in Ω. As T_n is continuous, we deduce from (5.23) and (5.24) that $T_n(u) = v_n \in H^1(\Omega)$. Appluying Lemma 3.7, we deduce that as $m \to \infty$,

$$\nabla_{\mathcal{M}_m} T_n(u_m) \rightharpoonup \nabla T_n(u) \text{ in } (L^2(\Omega))^d. \tag{5.25}$$

To end, we prove that $\text{med}\,(u) = 0$. Due to the point-wise convergence of u_m to u, the sequence $\mathbb{1}_{\{u_m>0\}}\mathbb{1}_{\{u>0\}}$ converges to $\mathbb{1}_{\{u>0\}}$ a.e. as $h_{\mathcal{M}}$ goes to zero. Using Fatou's lemma and the fact that $\underline{\text{med}}\,(u_m) = 0$, we have

$$\begin{aligned} \text{meas}\{u(x) > 0\} &\leq \liminf \int_\Omega \mathbb{1}_{\{u_m>0\}}\mathbb{1}_{\{u>0\}} dx \\ &\leq \liminf\ \text{meas}\{u_m(x) > 0\} \\ &\leq \frac{\text{meas}(\Omega)}{2}. \end{aligned}$$

Similarly, from the convergence of $\mathbb{1}_{\{u_m<0\}}\mathbb{1}_{\{u<0\}}$ to $\mathbb{1}_{\{u<0\}}$ a.e. as $m \to \infty$, we get

$$\text{meas}\{u(x) < 0\} \leq \frac{meas(\Omega)}{2}.$$

It follows that $0 \in \text{ med }(u)$. Since u is finite a.e., we deduce that for n large enough, $\text{med }(T_n(u)) = \text{ med }(u)$. Moreover, as $T_n(u)$ belongs to $H^1(\Omega)$, the median of u is unique and it is equal to 0. □

In the following proposition we prove a uniform estimate on the truncated energy of $u_{\mathcal{M}}$ (see (5.26)) which is crucial to pass to the limit in the approximate problem. We explicitly observe that (5.26) is the discrete version of (1.4) which is imposed in the definition of the renormalized solution for elliptic equation with L^1-data.

Proposition 5.4. ***(Discrete estimate on the energy)***
Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying (3.1) and such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. For any $m \geq 0$, let us consider $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ a solution to (3.6)-(3.8) and (3.11)-(3.12), and let u be a measurable function

finite a.e. in Ω such that, up to a subsequence still indexed by m, (5.16)-(5.19) hold. Then we have

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to 0}}\frac{1}{n}\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(u_K^m-u_L^m)(T_n(u_K^m)-T_n(u_L^m))=0 \tag{5.26}$$

where $u_L=0$ if $\sigma\in\mathcal{E}_{ext}$, and

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to 0}}\frac{1}{n}\sum_{\sigma=K/L\in\mathcal{E}_{int}}|\sigma||v_{K,\sigma}||u_{\sigma,+}^m||(T_n(u_{\sigma,+}^m)-T_n(u_{\sigma,-}^m)|=0. \tag{5.27}$$

Proof. We first establish (5.26). Let $m\geq 1$ and $u_m=(u_K^m)_{K\in\mathcal{T}_m}$ be a solution of (3.6)-(3.8) and (3.11)-(3.12). Multiplying each equation of the scheme by $\dfrac{T_n(u_K^m)}{n}$, summing on $K\in\mathcal{T}_m$ and gathering by edges lead to $T_1+T_2+T_3=T_4$ with

$$\begin{aligned}
T_1 &= \frac{1}{n}\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(u_K^m-u_L^m)(T_n(u_K^m)-T_n(u_L^m)),\\
T_2 &= \frac{1}{n}\sum_{\sigma=K/L\in\mathcal{E}_{int}}|\sigma|v_{K,\sigma}u_{\sigma,+}^m(T_n(u_K^m)-T_n(u_L^m)),\\
T_3 &= \frac{1}{n}\sum_{K\in\mathcal{T}}|K|b_Ku_K^mT_n(u_K^m),\\
T_4 &= \frac{1}{n}\sum_{K\in\mathcal{T}}\int_K fT_n(u_K^m)dx.
\end{aligned}$$

Since b is nonnegative and since $rT_n(r)\geq 0\,\forall r$, we get $T_3\geq 0$. Due to the definition of u_m we have $T_4=\displaystyle\int_\Omega f\frac{T_n(u_m)}{n}dx$. In view of the point-wise convergence of u_m to u, we obtain that $T_n(u_m)$ converges to $T_n(u)$ a.e. and weak * as $h_{\mathcal{M}_m}\longrightarrow 0$. It follows that $\displaystyle\lim_{h_{\mathcal{M}_m}\to 0}T_4=\int_\Omega f\frac{T_n(u)}{n}dx$. Since u is finite a.e. in Ω, $\dfrac{T_n(u)}{n}$ converges to 0 a.e. and in L^∞ weak, and since f belongs to $L^1(\Omega)$, the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to+\infty}\lim_{h_{\mathcal{M}_m}\to 0}T_4=0. \tag{5.28}$$

Using the same arguments as in [14] and the fact that $\underline{\text{med}}\,(T_n(u_m))=0$, we prove that

$$-T_2\leq\frac{1}{n}\frac{r^2d|v|^2_{L^2(\Omega)^d}}{2}+\frac{1}{2}T_1. \tag{5.29}$$

From (5.29), we deduce (5.26). By the same manage as in [14], we prove (5.27). □

The following corollary is useful to pass to the limit in the diffusion term.

Corollary 5.5. *(see [14]) Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying (3.1) and such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. For any $m \geq 0$, let us consider $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ a solution to (3.6)-(3.8) and (3.11)-(3.12). Then,*

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{M}_m}\to 0}} \sum_{\substack{\sigma=K/L\in\mathcal{E}_{int},\\ |u_K^m|\leq 2n,\\ |u_L^m>4n}} \frac{|\sigma|}{d_\sigma}|u_L^m| = 0. \tag{5.30}$$

6. Convergence results

We intend here to prove Theorem 3.9 that is the main result of this paper. Before proving Theorem 3.9, we recall the following convergence result (see [14]) concerning the function (h_n) defined, for any $n \geq 1$, by

$$h_n(s) = \begin{cases} 0, & \text{if } s \leq -2n; \\ \frac{s}{n}+2, & \text{if } -2n \leq s \leq -n, \\ 1, & \text{if } -n \leq s \leq n, \\ \frac{-s}{n}+2, & \text{if } n \leq s \leq 2n, \\ 0, & \text{if } s \geq 2n. \end{cases} \tag{6.1}$$

FIGURE 2. The function h_n

Lemma 6.1. *Let $(\mathcal{M}_m)_{m\geq 1}$ be a sequence of admissible meshes satisfying (3.1) and such that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. For any $m \geq 1$, let us consider $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ a solution to (3.6)-(3.8) and (3.11)-(3.12), and such that (5.16)-(5.19) hold. We define the function $\widetilde{h}_n^m$ over the diamonds by*

$$\forall\sigma = K/L \in \mathcal{E}_{int}, \quad \forall x \in D_\sigma, \quad \widetilde{h}_n^m(x) = \frac{h_n(x_K^m) + h_n(x_L^m)}{2},$$

then $\widetilde{h}_n^m \to h_n(u)$ in $L^q(\Omega)$, $\forall q \in [2, +\infty[$ where $h_{\mathcal{M}_m} \to 0$, where u is the limit of u_m.

We are now in position to prove Theorem 3.9.

Proof. **Proof of Theorem 3.9**. Let $\varphi \in C_c^\infty(\Omega)$ and h_n the function defined by (6.1). Let $m \geq 1$ and let $u_m = (u_K^m)_{K\in\mathcal{T}_m} \in X(\mathcal{T}_m)$ be a solution to (3.6)-(3.8) and (3.11)-(3.12). We denote by φ_m the function defined by $\varphi_K = \varphi(x_K)$ for all $K \in \mathcal{T}_m$. Multiplying each equation of the scheme by $\varphi(x_K)h_n(u_K^m)$ (which is a discrete version of the test function used in the renormalized formulation), summing over the control volumes and gathering by edges, we get $T_1+T_2+T_3 = T_4$

with

$$\begin{aligned}
T_1 &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(u_K^m - u_L^m)\big(\varphi(x_K)h_n(u_K^m) - \varphi(x_L)h_n(u_L^m)\big),\\
T_2 &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| v_{K,\sigma} u_{\sigma,+}^m \big(\varphi(x_K)h_n(u_K^m) - \varphi(x_L)h_n(u_L^m)\big),\\
T_3 &= \sum_{K\in\mathcal{T}_m} |K| b_K u_K^m \varphi(x_K) h_n(u_K^m),\\
T_4 &= \sum_{K\in\mathcal{T}_m} \int_K f\varphi(x_K)h_n(u_K^m).
\end{aligned}$$

As far as the term T_4 is concerned, by the regularity of φ, we have $\varphi_m \to \varphi$ uniformly on Ω when $h_{\mathcal{M}_m} \to 0$. We now pass to the limit as $h_{\mathcal{M}_m} \to 0$. Since $h_n(u_m) \to h_n(u)$ a.e and L^∞ weak $*$, $\varphi_m \to \varphi$ uniformly, $|f\varphi_m h_n(u_m)| \le C_\varphi |f| \in L^1(\Omega)$, the Lebesgue dominated convergence theorem ensures that

$$T_4 = \int_\Omega f\varphi_m h_n(u_m)\,dx \underset{h_{\mathcal{M}_m}\to 0}{\longrightarrow} \int_\Omega f\varphi h_n(u)\,dx. \tag{6.2}$$

In view of the definition of b_m, and since b belongs to $L^1(\Omega)$, $b_m = (b_K)_{K\in\mathcal{T}_m}$ converges to b in $L^1(\Omega)$ as $h_{\mathcal{M}_m} \to 0$. With already used arguments we can assert that

$$T_3 = \int_\Omega b_m T_{2n}(u_m)\varphi_m h_n(u_m)dx \underset{h_{\mathcal{M}_m}\to 0}{\longrightarrow} \int_\Omega b\, T_{2n}(u)\,\varphi\, h_n(u)\,dx. \tag{6.3}$$

We now study the convergence of the diffusion term. We write

$$\begin{aligned}
T_1 &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma}(u_K^m - u_L^m)\big(\varphi(x_K)h_n(u_K^m) - \varphi(x_L)h_n(u_L^m)\big)\\
&= T_{1,1} + T_{1,2}
\end{aligned}$$

with

$$\begin{aligned}
T_{1,1} &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma} h_n(u_K^m)(u_K^m - u_L^m)(\varphi(x_K) - \varphi(x_L)),\\
T_{1,2} &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma} \varphi(x_L)(u_K^m - u_L^m)(h_n(u_K^m) - h_n(u_L^m)).
\end{aligned}$$

According to the definition of h_n, we use (5.26) to obtain

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{M}_m}\to 0}} T_{1,2} = 0. \tag{6.4}$$

By the same manage as in the proof of Theorem 4.1 in [14], we prove that

$$\lim_{h_{\mathcal{M}_m}\to 0} T_{1,1} = \int_\Omega h_n(u)\nabla T_{4n}(u).\nabla\varphi dx. \tag{6.5}$$

For the convection term we have

$$\begin{aligned}
T_2 &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| v_{K,\sigma} u^m_{\sigma,+}\big(\varphi(x_K)h_n(u^m_K) - \varphi(x_L)h_n(u^m_L)\big)\\
&= \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}\geq 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+}\big(\varphi(x_K)h_n(u^m_{\sigma,+}) - \varphi(x_L)h_n(u^m_{\sigma,-})\big)\\
&\quad + \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}< 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+}\big(\varphi(x_K)h_n(u^m_{\sigma,+}) - \varphi(x_L)h_n(u^m_{\sigma,-})\big)\\
&= \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}\geq 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+} h_n(u^m_{\sigma,+})(\varphi(x_K) - \varphi(x_L))\\
&\quad + \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}\geq 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+} \varphi(x_L)(h_n(u^m_{\sigma,+}) - h_n(u^m_{\sigma,-}))\\
&\quad - \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}< 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+} h_n(u^m_{\sigma,+})(\varphi(x_L) - \varphi(x_K))\\
&\quad - \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}< 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+} \varphi(x_K)(h_n(u^m_{\sigma,+}) - h_n(u^m_{\sigma,-}))\\
&= T_{2,1} + T_{2,2} + T_{2,3}
\end{aligned}$$

with

$$\begin{aligned}
T_{2,1} &= \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| v_{K,\sigma} u^m_{\sigma,+} h_n(u^m_{\sigma,+})(\varphi(x_K) - \varphi(x_L)),\\
T_{2,2} &= \sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}\geq 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+} \varphi(x_L)(h_n(u^m_{\sigma,+}) - h_n(u^m_{\sigma,-})),\\
T_{2,3} &= -\sum_{\sigma=K/L\in\mathcal{E}_{int},\, v_{K,\sigma}< 0} |\sigma| v_{K,\sigma} u^m_{\sigma,+} \varphi(x_K)(h_n(u^m_{\sigma,+}) - h_n(u^m_{\sigma,-})).
\end{aligned}$$

Using the same arguments as in the proof of Theorem 4.1 in [14], we obtain

$$\lim_{n\to+\infty} \overline{\lim_{h_{\mathcal{M}_m}\to 0}} (T_{2,2} + T_{2,3}) = 0 \tag{6.6}$$

and

$$\lim_{h_{\mathcal{M}_m}\to 0} T_{2,1} = -\int_\Omega T_{2n}(u) h_n(u) v.\nabla\varphi dx. \tag{6.7}$$

We are now in position to pass to the limit as $h_{\mathcal{M}_m} \to 0$ in the scheme (3.6)-(3.8) and (3.11)-(3.12). Gathering equations (6.2) to (6.7), we can assert that

$$\begin{aligned}
\int_\Omega h_n(u)\,\nabla u\cdot\nabla\varphi dx - \int_\Omega u\, h_n(u)\, v\cdot\nabla\varphi dx + \int_\Omega b\, u\, h_n(u)\,\varphi dx&\\
- \int_\Omega f\,\varphi\, h_n(u) dx = \lim_{h_{\mathcal{M}_m}\to 0} T(n,\varphi)&
\end{aligned} \tag{6.8}$$

where $\lim_{h_{\mathcal{M}_m}\to 0} |T(n,\varphi)| \leq ||\varphi||_{L^\infty(\Omega)}\,\omega(n)$ with $\omega(n)\to 0$ as $n\to+\infty$.

The fact that $h_n(u)\,\nabla u \in (L^2(\Omega))^d$, $u\, h_n(u)\, v \in L^2(\Omega)$ and $f\, h_n(u) \in L^1(\Omega)$, by using a density argument, we get that (6.8) remains true for $\varphi \in L^\infty(\Omega)\cap H^1(\Omega)$. Let $h \in W^{1,\infty}(\mathbb{R})$ with supp $h \subset [-l,l], l>0$ and let $\psi \in L^\infty(\Omega)\cap H^1(\Omega)$. Taking $h(u)\psi$ as a test function in (6.8), we obtain

$$\begin{aligned}\Big|\int_\Omega \nabla u\, h_n(u)\, h(u)\, \nabla\psi\, dx + \int_\Omega \nabla u\, h_n(u)\, \psi\, \nabla u\, h'(u)\, dx \\ - \int_\Omega u\, h_n(u)\, h(u)\, v\cdot\nabla\psi\, dx - \int_\Omega u\, h_n(u)\, h'(u)\, \psi\, v\cdot\nabla u\, dx \\ + \int_\Omega b\, u\, h_n(u)\, h(u)\, \psi\, dx - \int_\Omega \psi\, h(u)\, h_n(u)\, f\, dx\Big| \\ \leq \|\varphi\|_{L^\infty(\Omega)}\omega(n).\end{aligned}$$

Passing to the limit as $n \to +\infty$ in the previous inequality yields that :

$$\begin{aligned}\int_\Omega \nabla u\, h(u)\, \nabla\psi\, dx \quad &+ \quad \int_\Omega \nabla u\, \psi\, \nabla u\, h'(u)\, dx \\ &- \quad \int_\Omega u\, h(u)\, v\cdot\nabla\psi\, dx - \int_\Omega u\, h'(u)\, \psi\, v\cdot\nabla u\, dx \\ &\qquad + \int_\Omega b\, u\, h(u)\, \psi\, dx = \int_\Omega f\, \psi\, h(u)\, dx,\end{aligned}$$

which is Equality (1.5) in the definition of a renormalized solution. It remains to prove that u satisfies the decay (1.4) of the truncate energy.
Thanks to the discrete estimate on the energy (5.26) we get,

$$\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to 0}}\frac{1}{n}\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(T_{2n}(u_K^m)-T_{2n}(u_L^m))^2 = 0$$

and

$$\begin{aligned}\sum_{\sigma=K/L\in\mathcal{E}_{int}}\frac{|\sigma|}{d_\sigma}(T_{2n}(u_K^m)-T_{2n}(u_L^m))^2 \quad &= \quad \sum_{\sigma=K/L\in\mathcal{E}_{int}}|\sigma|d_\sigma\left(\frac{T_{2n}(u_K^m)-T_{2n}(u_L^m)}{d_\sigma}\right)^2 \\ &= \quad \sum_{\sigma=K/L\in\mathcal{E}_{int}} d|D_\sigma|\left(\frac{T_{2n}(u_K^m)-T_{2n}(u_L^m)}{d_\sigma}\right)^2 \\ &= \quad \frac{1}{d}\sum_{\sigma=K/L\in\mathcal{E}_{int}}|D_\sigma|\left(d\frac{T_{2n}(u_K^m)-T_{2n}(u_L^m)}{d_\sigma}\right)^2 \\ &= \quad \frac{1}{d}\int_\Omega|\nabla_{\mathcal{M}_m}T_{2n}(u_m)|^2dx,\end{aligned}$$

hence, $\lim_{n\to+\infty}\overline{\lim_{h_{\mathcal{M}_m}\to 0}}\frac{1}{n}|\nabla_{\mathcal{M}_m}T_{2n}(u_m)|^2 = 0$. Since $\nabla_{\mathcal{M}_m}T_{2n}(u_m)$ converges weakly in $L^2(\Omega)^d$, we have also

$$\frac{1}{n}\int_\Omega|\nabla T_{2n}(u)|^2\, dx \leq \liminf_{h_{\mathcal{M}_m}\to 0}\frac{1}{n}\int_\Omega|\nabla_{\mathcal{M}_m}T_{2n}(u_m)|^2 dx,$$

which leads to

$$\lim_{n\to 0}\frac{1}{n}\int_\Omega |\nabla T_{2n}(u)|^2\,dx = 0.$$

Since the renormalized solution u is unique and $T_n(u) \in H^1(\Omega)$ for any $n > 0$, we conclude that the whole sequence $u_{\mathcal{M}_m}$ converges to u in the sense that for all $n > 0$, $T_n(u_{\mathcal{M}_m})$ converges weakly to $T_n(u)$ in $H^1(\Omega)$, with $\operatorname{med} u = 0$. □

Conclusion

In this work, we showed that the approximate solution, by the finite volumes method, converges to the renormalized solution of convective-diffusive elliptic problem with Neumann boundary conditions and L^1-data. Firstly, we recalled formulas and are given some notations and properties on Partial Differentials Equations. Secondly part we recalled the bases principle of the main methods of discretization, more precisely, the finite volume method. Thirdly, we studied a noncoercive elliptic convection-diffusion equation with Neumann boundary conditions and L^1-data. By adapting the strategy developed in the finite volume method, we proved the existence and the uniqueness of the discrete solution, and next, we proved that this approximate solution converges to the unique renormalized solution.

A. Appendix

We begin by recalling functions of bounded variation. Let Ω be an open set of $\mathbb{R}^d$ and $u \in L^1(\Omega)$. The total variation of u in Ω, denoted by $TV_\Omega(u)$, is defined by

$$TV_\Omega(u) = \sup\left\{\int_\Omega u(x)\ \operatorname{div}(\phi(x))\,dx,\ \phi \in C_c^1(\Omega),\ |\phi(x)| \le 1,\ \forall x \in \Omega\right\}.$$

The space $BV(\Omega)$ defined by

$$BV(\Omega) = \left\{u \in L^1(\Omega);\ TV_\Omega(u) < +\infty\right\},$$

is endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + TV_\Omega(u).$$

The reader can see [20] for more details about theses functions.
We have the following continuous embedding of $BV(\Omega)$ into $L^{\frac{d}{d-1}}(\Omega)$ for Lipschitz bounded connected domain Ω of $\mathbb{R}^d$, $d \ge 2$.

Theorem A.1. *(see [20]). There exists a constant $C(\Omega) > 0$ only depending on Ω such that, for all $u \in BV(\Omega)$,*

$$\left(\int_\Omega |u-c|^{\frac{d}{d-1}}\,dx\right)^{\frac{d-1}{d}} \le C(\Omega)\,TV_\Omega(u), \tag{A.1}$$

where $c \in med\ (u)$.

Following the idea of [2] in the proof of discrete Poincaré-Wirtinger median inequality, we prove the following.

Proposition A.2. (*Discrete Poincaré-Wirtinger median inequality*). *Let Ω be an open bounded connected polyhedral domain of $\mathbb{R}^d$ and let $\mathcal{M}$ be an admissible mesh satisfying* (3.1)*. Then for $1 \le p < +\infty$ there exists a constant $C > 0$ only depending on Ω, d and p such that*

$$\|u - c\|_{0,p} \le \frac{C}{\zeta^{(p-1)/p}} |u|_{1,p,\mathcal{M}}, \quad \forall u \in X(\mathcal{T}) \tag{A.2}$$

where c belongs to med(u).

Proof of Proposition A.2. Let $u = (u_K)_{K\in\mathcal{T}} \in X(\mathcal{T})$ and let $c \in \text{med}\,(u)$. We define $z \in X(\mathcal{T})$ by $z_K = (u_K - c)|u_K - c|^{p-1}$, $\forall K \in \mathcal{T}$. Using inequality (A.1) and the fact that $0 \in \text{med}\,(z)$, we obtain

$$\|z\|_{0,\frac{d}{d-1}} \le C(\Omega)|z|_{1,1,\mathcal{M}}, \tag{A.3}$$

which in turn, after using the inclusion of $L^{\frac{d}{d-1}}(\Omega)$ into $L^1(\Omega)$, gives

$$\|z\|_{0,1} \le C(\Omega, d)|z|_{1,1,\mathcal{M}}. \tag{A.4}$$

We also have for all $K, L \in \mathcal{T}$,

$$\begin{aligned} |z_K - z_L| &= |u_K - u_L||z'(y_{LK})|, \qquad \forall y_{LK} \in [u_K, u_L], \\ &\le p|u_K - u_L||y_{LK} - c|^{p-1} \\ &\le p|u_K - u_L|\Big(|u_K - c|^{p-1} + |u_L - c|^{p-1}\Big). \end{aligned} \tag{A.5}$$

Combining (A.4) and (A.5), we get

$$\begin{aligned} \|z\|_{0,1} = \||u-c|^p\|_{0,1} = \|u-c\|_{0,p}^p & \\ \le C \sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| p |u_K - u_L| \Big(|u_K - c|^{p-1} + |u_L - c|^{p-1}\Big). & \end{aligned} \tag{A.6}$$

By Hölder's inequality, we have

$$\begin{aligned} \|u - c\|_{0,p}^p &\le pC \left(\sum_{\sigma=K/L\in\mathcal{E}_{int}} \frac{|\sigma|}{d_\sigma^{p-1}} |u_K - u_L|^p\right)^{\frac{1}{p}} \\ &\times \left(\sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| \Big(d_\sigma^{\frac{p-1}{p}}\Big)^{\frac{p}{p-1}} \Big(|u_K - c|^{p-1} + |u_L - c|^{p-1}\Big)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\ &\le pC|u-c|_{1,p,\mathcal{M}} \left(\sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| d_\sigma \frac{p}{p-1} \big(|u_K - c|^p + |u_L - c|^p\big)\right)^{\frac{p-1}{p}}. \end{aligned} \tag{A.7}$$

By (3.1), we deduce that

$$\sum_{\sigma=K/L\in\mathcal{E}_{int}} |\sigma| d_\sigma \le \frac{1}{\zeta} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| d(x_K, \sigma) = \frac{d}{\zeta} \sum_{K\in\mathcal{T}} |K|. \tag{A.8}$$

Applying a discrete integration by parts and using (A.8), we obtain

$$\begin{aligned}
\|u-c\|_{0,p}^{p} &\leq pC(\Omega)|u-c|_{1,p,\mathcal{M}}\Big(\frac{p}{p-1}\Big)^{\frac{p-1}{p}}\left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|d_\sigma|u_K-c|^p\right)^{\frac{p-1}{p}} \\
&\leq \frac{C(\Omega,p,d)}{\zeta^{\frac{p-1}{p}}}|u-c|_{1,p,\mathcal{M}}\,\|u-c\|_{0,p}^{p-1}, \qquad \text{(A.9)}
\end{aligned}$$

which implies the desired result (A.2).

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