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WEIGHTED RATIONALIZED TOEPLITZ HANKEL OPERATORS

RUCHIKA BATRA VERMA *

ABSTRACT. In this paper, we introduce and study the notion of the weighted Rationalized Toeplitz Hankel operators. We discuss some properties of these operators. Also the weighted Rationalized Toeplitz Hankel matrix is introduced and studied.

1. INTRODUCTION

The theory of Toeplitz operators have been a subject of investigations by many mathematicians around the globe. Along with these operators, many mathematicians came up with different generalizations of these operators like Slant Toeplitz and Slant Hankel operators etc. All these operators have plenty of applications in wavelet analysis [8], differential equations [4], prediction theory [5] etc. Initially the Toeplitz and Hankel, slant Toeplitz operators were introduced on the L^2 and H^2 spaces but during the past few years they have been defined on the generalized spaces like weighted squence spaces $L^2(\beta)$ and $H^2(\beta)$ and the weighted function spaces L^p_w and H^p_w.

The generalization of all kinds of Toeplitz, Hankel, Slant Toeplitz, Slant Hankel is given in [3] as the Rationalized Toeplitz Hankel operator on the space L^2. The set of all Rationalized Toeplitz Hankel operators is denoted by RTHO(L^2). It is proved [3] that a bounded linear operator R on the space L^2 is a Rationalized Toeplitz Hankel operator of order (k_1, k_2) where k_1 and k_2 are relatively prime integers, if and only if $M_{z^{k_2}}R = RM_{z^{k_1}}$.

Further if k_1 and k_2 are not relatively prime then if $d = g.c.d\,(k_1, k_2)$ i.e. let $k_1 = dn$ and $k_2 = dm$ then a bounded operator R on L^2 is Rationalized Toeplitz Hankel operator if and only if

$$R|\tilde{N}_i = W_m M_{\tilde{\varphi}_i} W_n|\tilde{N}_i$$

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* Corresponding author.
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for some $\tilde{\varphi}_i$ in $L^\infty$and for $i = 1, 2, ...d-1$

$$
\begin{aligned}
L^2 &= \tilde{N}_0 \oplus \tilde{N}_1 \oplus \cdots \oplus \tilde{N}_{d-1} \\
&where \\
\tilde{N}_0 &= N_0 \oplus N_1 \oplus \cdots \oplus N_{m-1} \\
\tilde{N}_1 &= N_m \oplus N_{m+1} \oplus \cdots \oplus N_{2m-1} \\
\vdots & \qquad\qquad \vdots \\
\tilde{N}_{d-1} &= N_{(d-1)m} \oplus N_{(d-1)m+1} \cdots \oplus N_{dm-1}
\end{aligned}
$$

$N_i =$ The closed linear span of $\{z^{k_1 t+1} : t \in \mathbb{Z}\}$

Motivated by the study on the generalized spaces as given in [[1],[2]], here in this paper, we introduce the notion of weighted Rationalized Toeplitz Hankel operators on the space $L^2(\beta)$. We begin with the following notations.

Let $\beta = (\beta_n)_{n\in Z}$ be a sequence of positive numbers with $\beta_0 = 1$, $0 < \left(\frac{\beta_n}{\beta_{n+1}}\right) \leq 1$ when $n \geq 0$ and $0 < \left(\frac{\beta_n}{\beta_{n-1}}\right) < 1$ when $n \leq 0$.

Consider the Space

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, a_n \in \mathbb{C} \right\}$$

and

$$||f||_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

Then $(L^2(\beta), ||.||_\beta)$ is a *Hilbert Space* with an orthonormal basis $\left\{ e_n(z) = \frac{z^n}{\beta_n} \right\}_{n\in\mathbb{Z}}$ and with respect to the inner product defined as

$$\left\langle \sum a_n z^n, \sum b_n z^n \right\rangle = \sum a_n \overline{b_n} \beta_n^2$$

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n : \phi L^2(\beta) \subseteq L^2(\beta) \text{ } and \text{ } \exists \text{ } c \in \mathbb{R} \text{ } such \text{ } that \text{ } ||\phi f||_\beta \leq c||f||_\beta \ \forall f \in L^2(\beta) \right\}$$

$L^\infty(\beta)$ is a **Banach Space** with a norm defined as

$$||\phi_\infty|| = inf \left\{ c \ : \ ||\phi f||_\beta \leq c||f||_\beta \ \forall \ f \in L^2(\beta) \right\}$$

For ϕ in $L^\infty(\beta)$, Let M_ϕ^β denote the Multiplication operator on space $L^2(\beta)$. Then

$$M_\phi^\beta(f) = \phi f \ \forall \ f \in L^2(\beta)$$

and

$$M_\phi^\beta(e_k(z)) = \sum_{n=-\infty}^{\infty} a_{n-k} \frac{\beta_n}{\beta_k} e_n(z) \ \forall \ k \in \mathbb{Z}$$

We begin with the following definition

Definition 1.1. For $k \neq 0$, Let $W_k^\beta \ : \ L^2(\beta) \to L^2(\beta)$ be defined as

$$W_k^\beta e_n(z) = \begin{cases} \frac{\beta_{n/k}}{\beta_n} e_{n/k}(z) & if\ n\ is\ divisible\ by\ k \\ 0 & otherwise \end{cases}$$

W_k^β is a bounded linear operator on $L^2(\beta)$.The adjoint of W_k^β denoted by $W_k^{\beta^*}$ is given by

$$W_k^{\beta^*} e_n(z) = \frac{\beta_n}{\beta_{kn}} e_{kn}(z)\ \forall\ n \in \mathbb{Z}$$

2. Properties of W_k^β

We start with the following properties of W_k^β for all non zero integers k.

Theorem 2.1. *If g(z) is an $L^2(\beta)$ function and for fixed integer $k \neq 0$, $\frac{\beta_{kn}}{\beta_n} \leq M < \infty$, then $g(z^k)$ is also an $L^2(\beta)$ function.*

Proof. Let

$$g(z) \ = \ \sum_{n=-\infty}^{\infty} a_n z^n,$$

then

$$||g||_\beta^2 = \sum |a_n|^2 \beta_n^2 < \infty$$

Now,

$$\begin{aligned} g(z^k) \ &= \ \sum_{n=-\infty}^{\infty} a_n z^{kn} \\ &= \ \sum_{n=-\infty}^{\infty} a_n \beta_{kn} e_{kn}(z) \end{aligned}$$

So,

$$\begin{aligned} ||g(z^k)||_\beta^2 \ &= \ \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_{kn}^2 \\ &= \ \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 \frac{\beta_{kn}^2}{\beta_n^2} \\ &\leq \ M^2 \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \end{aligned}$$

Hence, $g(z^k)$ is also $L^2(\beta)$ function. □

Theorem 2.2. *For all f in $L^2(\beta)$,*

(i) $W_k^\beta f \in L^2(\beta)$

(ii) $W_k^{\beta^*} f \in L^2(\beta)$ *if* $\frac{\beta_{kn}}{\beta_n}$ *is bounded.*

Proof. (i) Let $f(z) = \sum a_n z^n$ be in $L^2(\beta)$. Then

$$||f||_\beta^2 = \sum |a_n|^2 \beta_{n^2} < \infty$$

Now,

$$\begin{aligned} W_k^\beta(f) &= W_k^\beta\left(\sum a_n z^n\right) \\ &= W_k^\beta\left(\sum a_n \beta_n e_n(z)\right) \\ &= \sum a_n W_k^\beta(\beta_n e_n(z)) \\ &= \sum_{m=-\infty}^{\infty} a_{km}\beta_m e_m(z) \\ So,\ ||W_k^\beta(f)||_\beta^2 &= \sum_{m=-\infty}^{\infty} |a_{km}|^2 \beta_m^2 \\ &= \sum_{m=-\infty}^{\infty} |a_{km}|^2 \beta_{km}^2 \frac{\beta_m^2}{\beta_{km}^2} \\ &\leq \sum_{m=-\infty}^{\infty} |a_{km}|^2 \beta_{km}^2 \\ &\leq \sum_{m=-\infty}^{\infty} |a_{nk}|^2 \beta_{km}^2 + \sum_{n \neq km} |a_n|^2 \beta_n^2 \\ &= \sum_{m=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \end{aligned}$$

Thus $W_k^\beta f \in L^2(\beta)$ for all f in $L^2(\beta)$.

(ii)

$$\begin{aligned} W_k^{\beta^*} f(z) &= W_k^{\beta^*}\left(\sum a_n z^n\right) \\ &= W_k^{\beta^*}\left(\sum a_n \beta_n e_n(z)\right) \\ &= \sum a_n \beta_n \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \\ &= \sum a_n z^{kn} \frac{\beta_n^2}{\beta_{kn}^2} \\ &= f_k(z^k) \end{aligned}$$

$$\begin{aligned} where\ ,\ \|f_k(z)\|_\beta^2 &= \|\sum a_n z^{kn}\frac{\beta_n^2}{\beta_{kn}^2}\|_\beta^2 \\ &= \sum |a_n|^2\frac{\beta_n^2}{\beta_{kn}^4}\beta_n^2 \\ &\leq \sum |a_n|^2\beta_n^2 < \infty \end{aligned}$$

This completes the proof. □

Theorem 2.3. *For $k \neq 0$*

(i) $W_k^\beta(f(z^k)) = f(z)\ \forall\ f \in L^2(\beta)$

(ii) $W_k^\beta W_k^{\beta^*} f(z) = f_k(z)$ *where* $f_k(z) = \sum a_n \frac{\beta_n^2}{\beta_{kn}^2} z^n$

(iii) $W_k^{\beta^*} W_k^\beta f(z) = g(z^k)$ *where* $g(z) = \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^n$

Proof. (i)

$$\begin{aligned} W_k^\beta(f(z^k)) &= W_k^\beta\left(\sum a_n z^{kn}\right)\ ,\ f = \sum a_n z^n \\ &= \sum a_n W_k^\beta(\beta_{kn}e_{kn}(z)) \\ &= \sum a_n\beta_n e_n(z) \\ &= \sum a_n z^n \\ &= f(z) \end{aligned}$$

Hence, $W_k^\beta(f(z^k)) = f(z)\ \forall\ f \in L^2(\beta)$

(ii)

$$\begin{aligned} W_k^\beta W_k^{\beta^*} f(z) &= W_k^\beta f_k(z^k) = f_k(z) \\ where\ ,\ f_k(z) &= \sum a_n \frac{\beta_n^2}{\beta_{kn}^2} z^n \end{aligned}$$

(iii)

$$\begin{aligned} W_k^{\beta^*} W_k^\beta f(z) &= W_k\left(\sum a_{kn} z^n\right) \\ &= \sum a_{kn}\beta_n\frac{\beta_n e_{kn}(z)}{\beta_{kn}} \\ &= \sum a_{kn}\frac{\beta_n^2}{\beta_{kn}^2} z^{kn} \\ &= g(z^k) \end{aligned}$$

where $g(z) = \sum_{n=-\infty}^{\infty} a_{kn}\frac{\beta_n^2}{\beta_{kn}^2} z^n$

This completes the proof. □

Theorem 2.4. *For any non zero integer k ,*

$$M_z^\beta W_k^\beta = W_k^\beta M_{z^k}^\beta$$

Proof. If m is divisible by k then $m = ki$ for some $i \in \mathbb{Z}$. Then

$$\begin{aligned} M_z^\beta W_k^\beta e_m(z) &= M_z^\beta \frac{\beta_i}{\beta_{ki}} e_i(z) \\ &= \frac{\beta_{i+1}}{\beta_{ki}} e_{i+1}(z) \\ \textit{And } W_k^\beta M_{z^k}^\beta e_m(z) &= W_k^\beta M_{z^k}^\beta (e_{ki}(z)) \\ &= W_k^\beta \beta_k e_k(z) e_{ki}(z) \\ &= W_k^\beta \frac{\beta_{(i+1)k}}{\beta_{ik}} e_{(i+1)k}(z) \\ &= \frac{\beta_{(i+1)k}}{\beta_{ik}} \frac{\beta_{i+1}}{\beta_{(i+1)k}} e_{i+1}(z) \\ &= \frac{\beta_{i+1}}{\beta_{ik}} e_{i+1}(z) \end{aligned}$$

Thus we have

$$M_z^\beta W_k^\beta e_m(z) = W_k^\beta M_{z^k}^\beta e_m(z)\ ,\ for\ m = ki.$$

If m is not divisible by k then

$$M_z^\beta W_k^\beta e_m(z) = 0 = W_k^\beta M_{z^k}^\beta e_m(z)$$

Thus,

$$M_z^\beta W_k = W_k M_{z^k}^\beta$$

This completes the proof. □

3. Weighted Rationalized Toeplitz Hankel Operators

We begin with the following definition.

Definition 3.1. For non zero integers k_1 and k_2 , the weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2) , $R_\phi^\beta : L^2(\beta) \to L^2(\beta)$ is defined as

$$R_\phi^\beta(f) = W_{k_1}^\beta M_\phi^\beta W_{k_2}^{\beta^*}(f)\ \forall\ f \in L^2(\beta)$$

.

If $e_i(z) = \left\{\frac{z^i}{\beta_i}\right\}_{i\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\beta)$, then

$$\begin{aligned} R_\phi^\beta(e_j(z)) &= W_{k_1}^\beta M_\phi^\beta W_{k_2}^{\beta^*}(e_j(z)) \\ &= W_{k_1}^\beta M_\phi^\beta \left(\frac{\beta_j}{\beta_{k_2 j}} e_{k_2 j}(z)\right) \\ &= W_{k_1}^\beta \frac{\beta_j}{\beta_{k_2 j}} M_\phi^\beta (e_{k_2 j}(z)) \\ &= W_{k_1}^\beta \frac{\beta_j}{\beta_{k_2 j}} \sum_{n=-\infty}^{\infty} a_{n-k_2 j} \frac{\beta_n}{\beta_{k_2 j}} e_n(z) \end{aligned}$$

$$\begin{aligned}
&= \frac{\beta_j}{\beta_{k_2 j}} \sum_{n=-\infty}^{\infty} a_{n-k_2 j} \frac{\beta_n}{\beta_{k_2 j}} W_{k_1}^{\beta} e_n(z) \\
&= \frac{\beta_j}{\beta_{k_2 j}} \sum_{m=-\infty}^{\infty} a_{mk_1-k_2 j} \frac{\beta_m}{\beta_{k_2 j}} e_m(z) \\
&= \frac{\beta_j}{(\beta_{k_2 j})^2} \sum_{m=-\infty}^{\infty} a_{mk_1-k_2 j} \beta_m e_m(z) \\
That\ is\ ,\ R_{\phi}^{\beta}(e_j(z)) &= \frac{\beta_j}{\beta_{k_2 j}^2} \sum_{m=-\infty}^{\infty} a_{mk_1-k_2 j} \beta_m e_m(z)
\end{aligned}$$

So, R_ϕ is abounded linear operator on $L^2(\beta)$.

Observation : If $k_2 = 1$ and $k_1 \geq 2$. Then

$$\begin{aligned}
R_{\phi}^{\beta}(e_j(z)) &= \frac{\beta_j}{\beta_j^2} \sum a_{mk_1-j} \beta_m e_m(z) \\
&= \frac{1}{\beta_j} \sum_{m=-\infty}^{\infty} a_{mk_1-j} \beta_m e_m(z)
\end{aligned}$$

Thus, if $k_2 = 1$ and $k_1 \geq 2$, then the weighted Rationalized Toeplitz Hankel operator becomes the Generalized weighted Slant Toeplitz operator.

We see that if $R_{\phi}^{\beta *}$, is the adjoint of weighted Rationalized Toeplitz Hankel Operator R_{ϕ}^{β}, then we have

$$\begin{aligned}
< R_{\phi}^{\beta *} e_j, e_i > &= < e_j, R_{\phi}^{\beta} e_i > \\
&= \left\langle e_j, \sum_{m=-\infty}^{\infty} \frac{\beta_i}{\beta_{k_2 i}^2} a_{mk_1-k_2 i} \beta_m e_m(z) \right\rangle \\
&= \frac{\beta_i \beta_j}{\beta_{k_2 i}^2} \bar{a}_{k_1 j - k_2 i}
\end{aligned}$$

So, if we consider the matrix of R_{ϕ}^{β} with respect to the orthonormal basis of $L^2(\beta)$ then $(i,j)^{th}$ element α_{ij} of matrix of R_{ϕ}^{β} is given by

$$\begin{aligned}
< R_{\phi}^{\beta} e_j, e_i > &= \left\langle \frac{\beta_j}{\beta_{k_2 j}^2} \sum_{m=-\infty}^{\infty} a_{mk_1-k_2 j} \beta_m e_m(z), e_i \right\rangle \\
&= \frac{\beta_i \beta_j}{\beta_{k_2 j}^2} a_{k_1 i - k_2 j}
\end{aligned}$$

Hence, the matrix of R_{ϕ}^{β} with respect to the basis of $L^2(\beta)$ is given by

$$\left(\begin{array}{cc|ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \frac{\beta_{-1}\beta_{-1}}{\beta^2_{-k_2}}a_{-k_1+k_2} & \frac{\beta_{-1}\beta_0}{\beta^2_0}a_{-k_1} & \frac{\beta_{-1}\beta_1}{\beta^2_{k_2}}a_{-k_1-k_2} & \frac{\beta_{-1}\beta_2}{\beta^2_{2k_2}}a_{-k_1-2k_2} & \frac{\beta_{-1}\beta_3}{\beta^2_{3k}}a_{-k_1-3k_2} & \cdots \\
\hline
\cdots & \frac{\beta_0\beta_{-1}}{\beta^2_{-k_2}}a_{k_2} & \frac{\beta_0}{\beta_0}a_0 & \frac{\beta_0\beta_1}{\beta^2_{k_2}}a_{-k_2} & \frac{\beta_0\beta_2}{\beta^2_{2k_2}}a_{-2k_2} & \frac{\beta_0\beta_3}{\beta^2_{3k}}a_{-3k_2} & \cdots \\
\cdots & \frac{\beta_1\beta_{-1}}{\beta^2_{-k_2}}a_{k_1+k_2} & \frac{\beta_1\beta_0}{\beta^2_0}a_{k_1} & \frac{\beta_1\beta_1}{\beta^2_{k_2}}a_{k_1-k_2} & \frac{\beta_1\beta_2}{\beta^2_{2k_2}}a_{k_1-2k_2} & \frac{\beta_1\beta_3}{\beta^2_{3k_2}}a_{k_1-3k_2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & &
\end{array}\right)$$

So,

$$\begin{aligned}
\alpha_{ij} &= (i,j)^{th}\ element\ of\ matrix\ of\ R^{\beta}_{\phi} \\
&= \frac{\beta_i\beta_j}{\beta^2_{k_2j}}a_{k_1i-k_2j}
\end{aligned}$$

Remark : If $\beta_i = 1\ \forall\ i\ Then\ \alpha_{ij} = a_{k_1i-k_2j}$ which is the $(i,j)^{th}$ element of the matrix of R_{ϕ}, the Rationalized Toeplitz Hankel operator with respect to the basis of L^2.

We denote the set of all weighted Rationalized Toeplitz Hankel Operators on the space $L^2(\beta)$ by WRTHO($L^2(\beta)$).

Theorem 3.2. *The mapping $\phi \to R^{\beta}_{\phi}$ is linear and one to one.*

Proof. If ϕ_1 and ϕ_2 are in $L^{\infty}(\beta)$ and α and β are scalars, then

$$\begin{aligned}
R_{\alpha\phi_1+\beta\phi_2} &= W^{\beta}_{k_1}M^{\beta}_{\alpha\phi_1+\beta\phi_2}W^{\beta*}_{k_2} \\
&= W^{\beta}_{k_1}(\alpha M^{\beta}_{\phi_1} + \beta M^{\beta}_{\phi_2})W^{\beta*}_{k_2} \\
&= \alpha W^{\beta}_{k_1}M^{\beta}_{\phi_1}W^{\beta*}_{k_2} + \beta W^{\beta}_{k_1}M^{\beta}_{\phi_2}W^{\beta*}_{k_2} \\
&= \alpha R^{\beta}_{\phi_1} + \beta R^{\beta}_{\phi_2}
\end{aligned}$$

So, if $R^{\beta}_{\phi_1} - R^{\beta}_{\phi} = 0$, then $R^{\beta}_{\phi_1-\phi_2} = 0$ Hence,

$$R^{\beta}_{\phi_1-\phi_2}(e_n(z)) = 0\ \forall\ n \in \mathbb{Z}$$

$$\begin{aligned}
\Longrightarrow W^{\beta}_{k_1}M^{\beta}_{\phi_1-\phi_2}W^{\beta*}_{k_2}e_n(z) &= 0\ \forall\ n \in \mathbb{Z} \\
\Longrightarrow W^{\beta}_{k_1}(\phi_1-\phi_2)W^{\beta*}_{k_2}e_n(z) &= 0\ \forall\ n \in \mathbb{Z} \\
\Longrightarrow W^{\beta}_{k_1}(\phi_1-\phi_2)\frac{\beta_n}{\beta_{k_2n}}e_{k_2n}(z) &= 0\ \forall\ n \in \mathbb{Z}
\end{aligned}$$

As this is true for all $n \in \mathbb{Z}$. Hence, $\phi_1 - \phi_{2=0}$ and thus $\phi_1 = \phi_2$.
This completes the proof. □

Theorem 3.3. *If $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a sequence of positive numbers such that $\beta_0 = 1$, $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, $0 < \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$ and $\frac{\beta_{nk_2}}{\beta_n} = 1$ then a* *bounded operator R on $L^2(\beta)$ is a weighted Rationalized Toeplitz Hankel operator of order (k_1, k_2) for relatively prime integers k_1 and k_2 if and only if*

$$M^{\beta}_{z^{k_2}}R = RM^{\beta}_{z^{k_1}}$$

.

Proof. Let $R = R^{\beta}_{\phi}$ be a weighted Rationalized Toeplitz Hankel operator of order (k_1, k_2). Then $R = W^{\beta}_{k_1}M^{\beta}_{\phi}W^{\beta *}_{k_2}$
Consider,

$$\begin{aligned}
M^{\beta}_{z^{k_2}}R(e_j(z)) &= M^{\beta}_{z^{k_2}}W^{\beta}_{k_1}M^{\beta}_{\phi}W^{\beta *}_{k_2}(e_j(z)) \\
&= M^{\beta}_{z^{k_2}}W^{\beta}_{k_1}M^{\beta}_{\phi}\left(\frac{\beta_j}{\beta_{k_2 j}}e_{k_2 j}(z)\right) \\
&= M^{\beta}_{z^{k_2}}W^{\beta}_{k_1}\left(\frac{\beta_j}{\beta_{k_2 j}}M^{\beta}_{\phi}e_{k_2 j}(z)\right) \\
&= M^{\beta}_{z^{k_2}}W^{\beta}_{k_1}\left(\frac{\beta_j}{\beta_{k_2 j}}\sum_{n=-\infty}^{\infty} a_{n-k_2 j}\frac{\beta_n}{\beta_{k_2 j}}e_n(z)\right) \\
&= M^{\beta}_{z^{k_2}}\frac{\beta_j}{\beta^2_{k_2 j}}\sum_{n=-\infty}^{\infty} a_{n-k_2 j}\beta_n W_{k_1}(e_n(z)) \\
&= M^{\beta}_{z^{k_2}}\frac{\beta_j}{\beta^2_{k_2 j}}\sum_{m=-\infty}^{\infty} a_{k_1 m-k_2 j}\beta_m(e_m(z)) \\
&= \frac{\beta_j}{\beta^2_{k_2 j}}\sum_{m=-\infty}^{\infty} a_{mk_1-k_2 j}\beta_{m+k_2}e_{m+k_2}(z) \\
&= \frac{\beta_j}{\beta^2_{k_2 j}}\sum_{m=-\infty}^{\infty} a_{mk_1-k_2(k_1+i)}\beta_m e_m(z)
\end{aligned}$$

On the other hand

$$\begin{aligned}
R_{\phi}M^{\beta}_{z^{k_1}}(e_j(z)) &= R_{\phi}M^{\beta}_{z^{k_1}}e_j(z) \\
&= R_{\phi}\left(\frac{\beta_{j+k_1}}{\beta_j}e_{j+k_1}(z)\right) \\
&= \frac{\beta_{j+k_1}}{\beta_j}R_{\phi}(e_{j+k_1}(z)) \\
&= \frac{\beta_{j+k_1}}{\beta_j}\sum_{m=-\infty}^{\infty} a_{mk_1-k_2(k_1+i)}\frac{\beta_{j+k}}{\beta^2_{k_2(j+k)}}\beta_m e_m(z) \\
&= \frac{\beta^2_{j+k_1}}{\beta_j\beta^2_{k_2(j+k_1)}}\sum_{m=-\infty}^{\infty} a_{mk_1-k_2(k_1+i)}\beta_m e_m(z)
\end{aligned}$$

From here, we can conclude that

$$M^{\beta}_{z^{k_2}} R_\phi(e_j(z)) = \left(\frac{\beta_j}{\beta^2_{k_j}}\right)\left(\frac{\beta_j\beta^2_{k_2(j+k_1)}}{\beta^2_{j+k_1}}\right) R_\phi M^{\beta}_{z^{k_1}}(e_j(z))$$

This is true for all j. That is

$$\begin{aligned} M^{\beta}_{z^{k_2}} R_\phi(e_j(z)) &= \left(\frac{\beta^2_j\beta^2_{k_2(j+k_1)}}{\beta^2_{k_2 j}\beta^2_{j+k_1}}\right) R_\phi M^{\beta}_{z^{k_1}}(e_j(z)) \\ &= R_\phi M^{\beta}_{z^{k_1}}(e_j(z)) \end{aligned}$$

This is true for all $j \in \mathbb{Z}$, as $\left(\frac{\beta_{nk_2}}{\beta_n} = 1\right)$

$$M^{\beta}_{z^{k_2}} R_\phi = R_\phi M^{\beta}_{z^{k_1}}$$

Conversely, if we suppose that a bounded linear operator R on the space $L^2(\beta)$ satisfies the condition $M^{\beta}_{z^{k_2}} R = R M^{\beta}_{z^{k_2}}$ then as we have proved in the converse part of Theorem 3.1[3], we can similarly proceed and prove that

$$R = W_{k_1} M^{\beta}_{\phi} W_{k_2^*}$$

This completes the proof. □

4. Weighted rationalized Toeplitz Hankel matrix

We have seen that the $(i,j)^{th}$ element of the matrix of R^{β}_{ϕ} with respect to the orthonormal basis of $L^2(\beta)$ is given by

$$\begin{aligned} \lambda_{ij} &= < R^{\beta}_{\phi} e_j, e_i > \\ &= \frac{\beta_i\beta_j}{\beta^2_{k_2 j}} a_{k_1 i - k_2 j} \end{aligned}$$

where $\phi(z) = \sum a_n z^n$ is the Fourier expansion of the function ϕ. This motivates us to define the following.

Definition 4.1. For relatively prime integers k_1 and k_2, we define the Weighted Rationalized Toeplitz Hankel matrix corresponding to the weight sequence (β_n) as a doubly infinite matrix (λ_{ij}) such that

$$\lambda_{i+k_2, j+k_1} = \frac{\beta_{i+k_2}}{\beta_i}\frac{\beta_{k_2 j}}{\beta_{k_2(j+k_1)}}\lambda_{i,j}\ ,\ i,j = 0, \pm1, \pm2, \cdots$$

.

OBSERVATIONS:

(1) If $k_1 = k_2 = 1$ then

$$\lambda_{ij} = \frac{\beta_i\beta_j}{\beta^2_j} a_{i-j} = \frac{\beta_i}{\beta_j} a_{i-j}$$

which is weighted Toeplitz matrix as

$$\lambda_{i+1,j+1} = \frac{\beta_{i+1}}{\beta_i}\frac{\beta_j}{\beta_{j+1}}\lambda_{ij}$$

(2) If $k_1 = 2$ and $k_2 = 1$ then

$$\lambda_{ij} = \frac{\beta_i\beta_j}{\beta_j^2}a_{2i-j} = \frac{\beta_i}{\beta_j}a_{2i-j}$$

which is weighted Slant Toeplitz matrix as

$$\lambda_{i+1,j+2} = \frac{\beta_{i+1}\beta_j}{\beta_i\beta_{j+2}}\lambda_{ij}$$

(3) For $k_1 \geq 2$ and $k_2 = 1$ we have

$$\lambda_{ij} = \frac{\beta_i\beta_j}{\beta_j^2}a_{ki-j} = \frac{\beta_i}{\beta_j}a_{ki-j}$$

which is a generalized weighted Slant Toeplitz matrix as

$$\lambda_{i+1,j+k} = \frac{\beta_{i+1}}{\beta_i}\frac{\beta_j}{\beta_{j+k}}\lambda_{ij}$$

As we have done in the previous section that for the section (β_n) if for all n $\frac{\beta_{k_2 n}}{\beta_n} = 1$, then a bounded operator R on $L^2(\beta)$ is a Weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2) if and only if $M^{\beta}_{z^{k_2}}R = RM^{\beta}_{z^{k_1}}$

Here, now we give characterization of Weighted Rationalized Toeplitz Hankel Operator in terms of Weighted Rationalized Toeplitz Hankel matrix (under the same conditions of the above said theorem).

Theorem 4.2. *A necessary and sufficient condition that an operator R^{β} on $L^2(\beta)$ is a Weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2) [where k_1 and k_2 are relatively prime integers] is that its matrix with respect to the orthonormal basis $\left\{e_n(z) = \frac{z^n}{\beta_n}\right\}_{n\in\mathbb{Z}}$ is a Weighted Rationalized Toeplitz Hankel matrix of same order.*

Proof. Suppose that R^{β} is a Weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2). Then as we have seen the corresponding matrix (λ_{ij}) of R^{β} w.r.t the orthonormal basis $\left(e_n(z) = \frac{z^n}{\beta_n}\right)_{n\in\mathbb{N}}$ of $L^2(\beta)$ is given by

$$\begin{aligned}\lambda_{i,j} &= < R^{\beta}e_j, e_i > \\ &= \frac{\beta_i\beta_j}{\beta_{k_2 j}^2}a_{k_1 i - k_2 j}\end{aligned}$$

Further for relatively prime integers k_1 and k_2

$$\begin{aligned}\lambda_{i+k_2,j+k_1} &= < R^{\beta}e_{j+k_1}, e_{i+k_2} > \\ &= \frac{\beta_{i+k_2}\beta_{j+k_1}}{\beta_{k_2(j+k_1)}^2}a_{k_1(i+k_2)-k_2(j+k_1)} \\ &= \frac{\beta_{i+k_2}\beta_{j+k_1}}{\beta_{k_2(j+k_1)}^2}a_{k_1 i - k_2 j}\end{aligned}$$

$$= \frac{\beta_{i+k_2}\beta_{j+k_1}}{\beta^2_{k_2(j+k_1)}}\frac{\beta^2_{k_2j}}{\beta_i\beta_j}\lambda_{ij}$$

Hence,

$$\lambda_{i+k_2,j+k_1} = \frac{\beta_{i+k_2}}{\beta_i}\frac{\beta_{k_2j}}{\beta_{k_2(j+k_1)}}\lambda_{ij} \quad \left(as\ \frac{\beta_{k_2n}}{\beta_n} = 1\right)$$

Hence, the matrix of R^β is a Weighted Rationalized Toeplitz Hankel matrix. Conversely, Let the matrix of R^β_ϕ be the Weighted Rationalized Toeplitz Hankel matrix. Then

$$< R^\beta e_{j+k_1}, e_{i+k_2} >= \frac{\beta_{i+k_2}}{\beta_i}\frac{\beta_{k_2j}}{\beta_{k_2(j+k_1)}} < R^\beta e_j, e_i >$$

Now,

$$\begin{aligned}
< R^\beta M^\beta_{z^{k_1}} e_j, e_i > &= < R^\beta \frac{\beta_{j+k_1}}{\beta_j} e_{j+k_1}, e_i > \\
&= \frac{\beta_{j+k_1}}{\beta_j} < R^\beta e_{j+k_1}, e_i > \\
&= \frac{\beta_{j+k_1}}{\beta_j}\frac{\beta_{k_2j}}{\beta_{k_2(j+k_1)}}\frac{\beta_i}{\beta_{i-k_2}} < R^\beta e_j, e_{i-k_2} > \\
&= \frac{\beta_i}{\beta_{i-k_2}} < R^\beta e_j, e_{i-k_2} > \\
&= < R^\beta e_j, \frac{\beta_i}{\beta_{i-k_2}} e_{i-k_2} > \\
&= < R^\beta e_j, M^{\beta *}_{z^{k_2}} e_i > \\
&= < M^\beta_{z^{k_2}} R^\beta e_j, e_i >
\end{aligned}$$

This is true $\forall\ i, j \in \mathbb{Z}$.
Thus we conclude that

$$M^\beta_{z^{k_2}} R^\beta = R^\beta M^\beta_{z^{k_1}}$$

and hence R^β is a Rationalized Toeplitz Hankel Operator.

□

References

[1] S.C. Arora, R. Kathuria *On Generalized Slant Weighted Toeplitz operators*, World Academy of Science,Engineering and Technology **75** (2011), 1103-1106.

[2] S.C. Arora, R. Kathuria, *On kth order Slant Toeplitz operators* , The Scientific World Journal, **2013** (2013), Article ID 960853,Pages 5.

[3] R. Batra, *Rationalization of Toeplitz Hankel operators*, Gulf Journal of Mathematics, **12(2)** (2022), 86-96.

[4] T. Goodman,C.Micchelli and J.Ward *Spectral radius formula for subdivision operators*, Recent Advances in wavelet Analysis,Academic Press,L.Schumaker and G.Webb,Eds, (1994), 335-360.

[5] H.Helson and G.Szego , *A problem in prediction theory*,Annali di Mathematica Pura ed Applicata, **51(1)** (1960), 107-138.

[6] M.C. Ho, *Properties of Slant Toeplitz operators*, Indiana Univ. Math. J, **45(3)** (1996), 843-862.

[7] P.R.Halmos, *A Hilbert space Problem Book*, Springer Verlag, News York, 1979 .
[8] L.Villemoes, *Wavelet analysis of refinement equations*,SIAM Journal on Mathematical Analysis, **25** (1994), 1433-1460.

(R. Batra Verma) , Ramjas College, University of Delhi, Delhi-110007, India.
Email address: ruchikaverma@ramjas.du.ac.in