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Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 14 (2023) 55-63

# RESULTS OF SEMIGROUP OF LINEAR OPERATORS GENERATING A SEMILINEAR EVOLUTION EQUATION IN $\mathbb{R}^3$

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ABSTRACT. In this paper, we present results of $\omega$-order preserving partial contraction mapping generating a semilinear evolution equation in $\mathbb{R}^3$. We consider a bounded domain $\Omega$ with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$ and use the technique of semilinear equations with analytic semigroups to obtain a strong solution of the initial value problem. We showed that the operator $A$ is symmetric and that $A$ is the infinitesimal generator of a $C_0$-semigroup on $\mathcal{L}^2(\Omega)$. Finally we established that operator $A$ is well defined, self adjoint and its the infinitesimal generator of an analytic semigroup.

## 1. INTRODUCTION

Consider the following nonlinear initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{3} u \frac{\partial u}{\partial x_i} & in\ (0, T] \times \Omega \\ u(t, x) - 0 & on\ [0, T] \times \partial\Omega \\ u(0, x) = u_0(x) & \Omega. \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$ and we denote by $(\cdot)$ and $\|\cdot\|$ the scalar product and norm in $\mathcal{L}^2(\Omega)$.

We define an operator $A$ by

$$D(A) = H^2(\Omega) \cap H_0'(\Omega),\ Au = -\Delta u \tag{1.2}$$

for $u \in D(A)$ and $A \in \omega - OCP_n$.

In particular, we have for some $\delta > 0$,

$$(Au, u) = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}u) = \|A^{\frac{1}{2}}u\|^2 = \|\nabla u\|^2 \geqq \delta\|u\|^2 \tag{1.3}$$

where $\nabla u$ is the gradient of $u$ and the inequality is a consequence of Poincare's inequality. The domain of $A$ consists of Hölder continuous unctions.

Assume $X$ is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the $\omega$-order preserving partial contraction mapping, $M_m$ be a matrix, $L(X)$ be a bounded linear

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*Date*: Received: Jan 19, 2023; Accepted: Apr 28, 2023.
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2010 *Mathematics Subject Classification.* Primary 06F15, 06F05; Secondary 20M05.
*Key words and phrases.* $\omega$-$OCP_n$, Evolution Equation, $C_0$-semigroup, Analytic Semigroup.

operator on $X$, $P_n$ a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of $A$. This paper consist of results of $\omega$-order preserving partial contraction mapping generating a Semilinear evolution equation in $\mathbb{R}^3$. Akinyele *et al.* [1], characterized $\omega$-order reversing partial contraction mapping as a compact semigroup of linear operator and also in [2], Akinyele *et al.*, obtained differentiable and analytic results on $\omega$-order preserving partial contraction mapping in semigroup of linear operator. Balakrishnan [3], presented an operator calculus for infinitesimal generators of semigroup. Banach [4], established and introduced the concept of Banach spaces. Brezis and Gallouet [5], generated nonlinear Schrödinger evolution equation. Chill and Tomilov [6], presented some resolvent approach to stability operator semigroup. Davies [7], obtained linear operators and their spectra. Engel and Nagel [8], introduced one-parameter semigroup for linear evolution equations. Omosowon *et al.* [9], generated some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [10], Omosowon *et al.*, introduced dual Properties of $\omega$-order reversing partial contraction mapping in semigroup of linear operator. Omosowon *et al.* [11], established a regular weak*-continuous semigroup of linear operators, and also in [12], Omosowon *et al.*, obtained a quasilinear equations of evolution on semigroup of linear operator.reversing partial contraction mapping generating a differential operator. Omosowon *et al.* [13], deduced results of semigroup of linear equation generating a wave equation. Pazy [14], presented asymptotic behavior of the solution of an abstract evolution and some applications and also in [15], obtained a class of semi-linear equations of evolution. Rauf and Akinyele [16], introduced $\omega$-order preserving partial contraction mapping and obtained its properties, also in [17], Rauf *et al.*, established some results of stability and spectra properties on semigroup of linear operator. Vrabie [18], proved some results of $C_0$-semigroup and its applications. Yosida [19], deduced some results on differentiability and representation of one-parameter semigroup of linear operators.

## 2. Preliminaries

**Definition 2.1** ($C_0$-Semigroup) [18]
A $C_0$-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

**Definition 2.2** ($\omega$-$OCP_n$) [16]
A transformation $\alpha \in P_n$ is called $\omega$-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

**Definition 2.3** (Evolution Equation) [14]
An evolution equation is an equation that can be interpreted as the differential law of the development (evolution) in time of a system. The class of evolution equations includes, first of all, ordinary differential equations and systems of the form

$$u' = f(t,u), u'' = f(t,u,u')$$

etc., in the case where u(t) can be regarded naturally as the solution of the Cauchy problem; these equations describe the evolution of systems with finitely many degrees of freedom.

**Definition 2.4** ( Mild Solution) [15]
A continuous solution $u$ of the integral equation.

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds \tag{2.1}$$

will be called a mild solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t,u(t)),\ t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{2.2}$$

if the solution is a Lipschitz continuous function.

**Definition 2.5**(Analytic Semigroup) [18]
We say that a $C_0$-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:
(i) $T(t) = S(t)$ for each $t \geq 0$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;
(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the $C_0$-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

**Example 1**

For every $2 \times 2$ matrix in $[M_m(\mathbb{R}^n)]$.
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then we have

$$e^{tA} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

**Example 2**
For every $3 \times 3$ matrix in $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on $X$.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then we have

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

**Example 3**

Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\|\cdot\|_\infty$ and let $\{T(t); t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[T(t)f](s) = f(t+s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{T(t); t \in \mathbb{R}_+\}$ satisfies Examples 1 and 2 above.

## 3. Main Results

This section present results of semigroup of linear operator by using $\omega$-$OCP_n$ to generates a Semilinear evolution equation in $\mathbb{R}^3$:

**Theorem 3.1**

Assume $A : D(A) \subseteq H^2(\Omega) \to H^2(\Omega)$ is the infinitesimal generator of analytic semigroup $\{T(t) : t \geq 0\}$. $D(A)$ consists of Holder continuous functions with exponents $\frac{1}{2}$ and there is a constant $C$ such that

$$|u(x_1) - u(x_2)| \leq C\|Au\||x_1 - x_2|^{\frac{1}{2}} \quad for \quad u \in D(A) \tag{3.1}$$

where $A \in \omega - OCP_n$, $x_i \in \mathbb{R}^3$ and $|x_1 - x_2|$ denotes the Euclidean distance between $x_1$ and $x_2$.

**Proof:**

For $\varphi \in C_0^\infty(\Omega)$ we have the classical identity

$$\varphi(x) = C \int_\Omega \frac{\Delta\varphi(y)}{|x-y|} dy. \tag{3.2}$$

From (3.2) and the Cauchy-Schwartz inequality we deduce

$$\begin{aligned} |\varphi(x_1) - \varphi(x_2)|^2 &\leq C^2 \left( \int_\Omega \Delta\varphi(y) \left( \frac{1}{|x_1 - y|} - \frac{1}{|x_2 - y|} \right) dy \right)^2 \\ &\leq C^2 \int_\Omega |\Delta\varphi|^2 dy \int_\Omega \left( \frac{1}{|x_1 - y|} - \frac{1}{|x_2 - y|} \right)^2 dy \end{aligned}$$

but

$$\int_\Omega \left( \frac{1}{|x_1 - y|} - \frac{1}{|x_2 - y|} \right)^2 dy \leq C|x_1 - x_2|$$

where $C$ is a constant depending only on $\Omega$. Therefore,

$$|\varphi(x_1) - \varphi(x_2)| \leq C\|A\varphi\||x_1 - x_2|^{\frac{1}{2}}. \tag{3.3}$$

Approximating $u \in D(A)$ in $H^2(\Omega) \cap H_0'(\Omega)$ by a sequence $\varphi_n \in C_0^\infty(\Omega)$ and passing to the limit yields (3.1) since $H^2(\Omega) \subset C(\overline{\Omega})$ by letting $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ (e.g, $\partial\Omega$ is of class $C^m$), then

$$W^{k,p}(\Omega) \subset \mathcal{L}^{np/(n-kp)}(\Omega) \quad for \quad kp < n \tag{3.4}$$

and

$$W^{k,p}(\Omega) \subset C^m(\overline{\Omega}) \quad for \quad 0 \leq m < k - \frac{n}{p}. \tag{3.5}$$

Moreover, there exist constants $C_1$ and $C_2$ such that for any $u \in W^{m \cdot p}(\Omega)$, we have

$$\|u\|_{0,np/(n-kp)} \leq C_1 \|u\|_{k,p} \quad for \quad kp < n \tag{3.6}$$

and

$$\sup\left\{|D^\alpha u(x)| : |\alpha| \leq m, x \in \overline{\Omega} \leq C_2\|u\|_{k,p} \; for \; 0 \leq m < k - \frac{n}{p}\right\}. \tag{3.7}$$

The space $W_0^{k,p}$ is the subspace of elements of $W^{k,p}(\Omega)$ which vanish in some generalized sense on $\partial\Omega$. It can be shown that if $\partial\Omega$ is of class $C^k$ and $u \in C^{k-1}(\overline{\Omega}) \cap W_0^{k,p}(\Omega)$, then $u$ and its first $k-1$ normal derivatives vanish on $\partial\Omega$. Conversely, if $u \in C^{k-1}(\overline{\Omega}) \cap W_0^{k,p}(\Omega)$, then $u$ and its first $k-1$ normal derivatives vanish on $\partial\Omega$ then $u \in W_0^{k,p}(\Omega)$ and this achieved the proof.

**Theorem 3.2**
Suppose $A : D(A) \subseteq \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ is the infinitesimal generator of a semigroup $\{T(t) : t \geq 0\}$.Then there is a constant $C$ such that

$$\|u\|_{0,\infty}^4 \leq C\|Au\|^3\|u\| \tag{3.8}$$

for $u \in D(A)$ and $A \in \omega - OCP_n$.
**Proof:**
First we note that by (3.4), (3.5), (3.6) and (3.7), $u \in D(A)$ is in $C(\overline{\Omega})$ and since $\partial\Omega$ is assumed to be smooth it also follows that $u$ vanishes on $\partial\Omega$. For $u \equiv 0$, (3.8) is trivial. Let $\|u\|_{0,\infty} = \mathcal{L} > 0$. From Theorem 3.1, we have

$$|x(x_1) - u(x_2))| \leq K|x_1 - x_2|^{\frac{1}{2}} \tag{3.9}$$

where $K = C\|Au\|$. Without loss of generality we assume that $|u(0)| = \mathcal{L}$ and let $B_R$ be an open ball of radius $R = (\mathcal{L}/K)^2$ around 0. In this ball we have

$$|u(x)| > |u(0)| - |u(x) - x(0)| \geq L - K|x|^{\frac{1}{2}} > L - K\frac{L}{K} = 0. \tag{3.10}$$

Since $u$ vanishes on $\partial\Omega$ we deduce from (3.10) that $B_R \subset \Omega$ and for $x \in B_R$ we have

$$|u(x)| \geq L - K|x|^{\frac{1}{2}}. \tag{3.11}$$

Now,

$$\begin{aligned}\|u\|^2 &\geq \int_{B_R} |u(x)|^2 dx \\ &\geq \int_{B_R} (L - K|x|^{\frac{1}{2}})^2 dx \\ &= 4\pi L^2 R^3 \int_0^1 (1-\eta^{\frac{1}{2}})^2 \eta^2 d\eta \\ &= CL^2R^3 \\ &= CL^3K^{-6}, \end{aligned} \tag{3.12}$$

therefore we have

$$\|u\|_{0,\infty}^4 \leq C\|Au\|^3\|u\|$$

for all $u \in D(A)$ and $A \in \omega - OCP_n$. Hence, the proof is completed.

**Theorem 3.3**
Let $A : D(A) \subseteq \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ be the infinitesimal generator of a $C_0$-semigroup $\{T(t)_{t\geq 0}\}$. For $\gamma > \frac{3}{4}$ there is a constant $C$ depending only on $\gamma$ and $\Omega$ such that

$$\|u\|_{0,\infty} \leq C\|A^\gamma u\| \tag{3.13}$$

for all $u \in D(A)$ and $A \in \omega - OCP_n$.
**Proof:**
Suppose $\frac{3}{4} < \gamma < 1$. If $\omega = A^\gamma u$, then (3.13) is equivalent to

$$\|A^{-\gamma}\omega\|_{0,\infty} \leq C\|\omega\|. \tag{3.14}$$

In order to estimate $\|A^{-\gamma}\omega\|_{0,\infty}$, we use

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(tI + A)^{-1} dt \quad 0 < \alpha < 1.$$

So that

$$A^{-\gamma}\omega = \frac{\sin \pi\gamma}{\pi} \int_0^\infty t^{-\gamma}(tI + A)^{-1}\omega dt. \tag{3.15}$$

From (1.3) it follows that $\|A^{-1}\| \leq \delta^{-1}$ and that for every $t \geq 0$ we have

$$\|(tI + A)^{-1}\omega\| \leq (t + \delta)^{-1}\|\omega\|. \tag{3.16}$$

Also since $A$ is dissipative in $\mathcal{L}^2(\Omega)$ we have

$$\|A(tI + A)^{-1}\omega\| \leq \|\omega\| \tag{3.17}$$

and since $(tI + A)^{-1}\omega \in D(A)$, then Theorem 3.2 yields

$$\|(tI + A)^{-1}\omega\|_{0,\infty}^4 \leq C\|A(tI + A)^{-1}\|^3\|(tI + A)^{-1}\omega\|. \tag{3.18}$$

Combining (3.15), (3.16), (3.17) and (3.18) yields

$$\|A^{-\gamma}\omega\|_{0,\infty} \leq C_1 \int_0^\infty t^{-\gamma}(\delta + 1)^{\frac{1}{4}}\|\omega\| dt. \tag{3.19}$$

For $\frac{3}{4} < \gamma < 1$, the integral in (3.19) converges and we have

$$\|A^{-\gamma}\omega\|_{0,\infty} \leq C\|\omega\|. \tag{3.20}$$

For $\gamma \geq 1$, the result follows from the results for $\frac{3}{4} < \gamma < 1$ via the estimate $\|A^{-1}\| \leq \delta^{-1}$ and this achieved the proof.

**Theorem 3.4**
Assume $A : D(A) \subseteq H^2(\Omega) \to H^2(\Omega)$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t)_{t\geq 0}\}$. Let

$$f(u) = \sum_{i=1}^{3} u\frac{\partial u}{\partial x_i}. \tag{3.21}$$

If $\gamma > \frac{3}{4}$, $u \in D(A)$ and $A \in \omega - OCP_n$, then $f(u)$ is well defined and

$$\|f(u)\| \leq C\|A^{\gamma}u\|\|A^{\frac{1}{2}}u\|. \tag{3.22}$$

If $u, v \in D(A)$ and $A \in \omega - OCP_n$, then

$$\|f(u) - f(v)\| \leq C(\|A^{\gamma}u\|\|A^{\frac{1}{2}}u - A^{\frac{1}{2}}v\| + \|A^{\frac{1}{2}}v\|\|A^{\gamma}u - A^{\gamma}v\|). \tag{3.23}$$

**Proof:**
Since $D(A) \subset H^2(\Omega)$, then it follows from Sobolev's theorem which states that if $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ of class $C^m$, then

$$W^{k,p}(\Omega) \subset \mathcal{L}^{np/(n-kp)}(\Omega) \quad for \quad kp < n \tag{3.24}$$

and

$$W^{k,p}(\Omega) \subset C^m(\overline{\Omega}) \quad for \quad 0 \leq m < K - \frac{n}{p}. \tag{3.25}$$

Moreover, there exist constants $C_1$ and $C_2$ such that for any $u \in W^{m,p}(\Omega)$,

$$\|u\|_{0,np/(n-kp)} \leq C_1\|u\|_{k,p} \quad for \quad kp < n \tag{3.26}$$

and

$$\sup\{|D^{\alpha}u(x)| : |\alpha| \leq m,\ x \in \overline{\Omega} \leq C_2\|u\|_{k,p}\} \tag{3.27}$$

for $0 \leq m < K - \frac{n}{p}$ and it follows that $u \in \mathcal{L}^{\infty}(\Omega)$ and therefore $f(u) \in \mathcal{L}^2(\Omega)$ is thus well-defined. Moreover, from Theorem 3.3 we have

$$\|f(u)\| \leq \|u\|_{0,\infty}\|\nabla u\| \leq C\|A^{\gamma}u\|\|\nabla u\| = C\|A^{\gamma}u\|\|A^{\frac{1}{2}}u\|.$$

Also

$$\begin{aligned}\|f(u) - f(v)\| &\leq \|u\|_{0,\infty}\|\nabla(u - v)\| + \|u - v\|_{0,\infty}\|\nabla u\| \\ &\leq C(\|A^{\gamma}u\|\|A^{\frac{1}{2}}u - A^{\frac{1}{2}^{v}}v\| + \|A^{\frac{1}{2}}v\|\|A^{\gamma}u - A^{\gamma}v\|).\end{aligned} \tag{3.28}$$

Hence the proof in completed.

**Conclusion**
In this paper, it has been established that $\omega$-order preserving partial contraction mapping generates some results of semilinear evolution equation in $\mathbb{R}^3$

**Acknowledgment**
The authors acknowledge the management of the university of Ilorin for providing

us with a suitable research laboratory and library to enable us carried out this research.

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