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COMPLEMENTARY DISTANCE POLYNOMIAL AND ENERGY OF JOIN OF TWO GRAPHS

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ABSTRACT. For a connected graph G with diameter D, the complementary distance matrix is defined as, $CD(G) = [c_{ij}]$ in which $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and 0 otherwise, where d_{ij} is distance between the vertices v_i and v_j. The CD-polynomial has been studied for the join of two regular graphs when both the graphs are of diameter less than or equal to 2. In the present work, we study CD-polynomial for join of any two graphs and hence construct a pair of CD-equienergetic graphs by joining a regular graph (which is among a pair of CD-equienergetic graphs of same order and degree) with a non-regular graph. Also, CD-eigenvalues for these structures are studied interms of adjacency eigenvalues of G_1 and G_2 when both of them are regular.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper we consider simple, finite, connected and undirected graphs. A graph G is a pair of sets $\big(V(G), E(G)\big)$, where $V(G)$ is a non empty set of elements called vertices, and $E(G)$ is a set of elements called edges. Two vertices that are joined by an edge are said to be adjacenct or neighbours, the degree of a vertex counts to the number of neighbours, G is regular if all of its vertices have same degree. A (v_1, v_k)–path in G is an alternating sequence $[v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k]$ of vertices and edges from G in which all vertices and edges are distinct, the distance d_{ij} between two vertices v_i and v_j is the minimal length of the path connecting them, the maximal distance between the vertices is the diameter (D) of G. The complement $\overline{G}$ of G is the graph with vertex set same as G in which two vertices are adjacent if and only if they are not adjacent in G. The join of two disjoint graphs G_1 and G_2 denoted by $G_1 \bigtriangledown G_2$ is the graph obtained by taking a copy of G_1 and a copy of G_2, joining each vertex of G_1 to each vertex of G_2. If G is a graph on n vertices (of order n), the adjacency matrix is defied as $A(G) = [a_{ij}]$ in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and 0 otherwise. The polynomial associated to $A(G)$ defined by, $\phi\big(A(G); x\big) =\mid xI_n - A(G) \mid$ is called the characteristic polynomial, the roots of the equation $\mid xI_n - A(G) \mid= 0$ are the eigenvalues (denote them by

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λ_i's), their collection is called the spectrum and sum of absolute values of these eigenvalues is the energy [3]. Two graphs are said to be equienergetic, if they have same energy. Cospectral graphs are obviously equieneregtic, non cospectral equienergetic graphs are the study of interest. If G is an r-regular graph on n vertices with the eigenvalues $\lambda_1 = r, \lambda_2, \dots, \lambda_n$ then the eigenvalues of $\overline{G}$ are $n - r - 1, -1 - \lambda_2, \dots, -1 - \lambda_n$. It is noted that, for regular graphs row sum is constant (for all graph matrices) and hence whole regular graph can be taken as a single partition in studying join of a non-regular graph and a regular graph. J denotes the matrix with all etries equal to 1, I denotes the identity matrix. For other graph theoretical definitions and notations, we follow [1, 2]

Due to the considerable interest in deriving the additional structural descriptors for QSPR and QSAR models, O. Ivanciuc et al. [4] defined the CD-matrix, which represents a new molecular graph metric derived from graph distances. CD-matrix, which is viewed as one of the prominent matix in graph theoretical matrices in chemistry [5]. The complementary distance matrix for a graph G is defined as, $CD(G) = [c_{ij}]$ in which $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and 0 otherwise. The polynomial $\phi\big(CD(G); x\big) =\mid xI_n - CD(G) \mid$ is the CD-polynomial, roots of the equation $\mid xI_n - CD(G) \mid= 0$ are the CD-eigenvalues (denote them by μ_i's, the maximum among them be denoted by μ_{max}), their collection is called the CD-spectrum and sum of absolute values of these CD-eigenvalues is the CD-energy ($\mathcal{E}_{CD}$). Two graphs of equal CD-energy are CD-equienergetic. CD-cospectral graphs are obviously CD-equieneregtic, non CD-cospectral but CD-equienergetic graphs are in scope.

In 2019, Ramane et al. [7] studied the join of two graphs and hence provided a method for the construction of non CD-cospectral but CD-equienergetic graphs by joining two regular graphs (one among two is taken from a pair of CD-equienergetic graphs of same order and degree). The work related to CD-equienergetic graphs can be seen in [6, 8].

The aim of present work is to study CD-polynomial for join of any two graphs and hence construct a pair of CD-equienergetic graphs by joining a regular graph (which is among a pair of CD-equienergetic graphs of same order and degree) with a non-regular graph. Also, to establish relation between the CD-eigenvalues of $G_1 \bigtriangledown G_2$ with adjacency eigenvalues of G_1 and G_2 when both of them are regular.

We need the following Proposition for our study.

Proposition 1.1. (Schur Complement [1]) *Suppose that the order of all four matrices D_{11}, D_{12}, D_{21} and D_{22} satisfy the rules of operations on matrices. Then, we have*

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \begin{cases} |D_{22}|\,|D_{11} - D_{12}D_{22}^{-1}D_{21}|, & \textit{if } D_{22} \textit{ is a non-singular matrix,} \\ |D_{11}|\,|D_{22} - D_{21}D_{11}^{-1}D_{12}|, & \textit{if } D_{11} \textit{ is a non-singular matrix.} \end{cases}$$

2. CD-polynomial for join of two graphs

This section focuses on the polynomial and eigenvalues associated with the CD-matrix of join of two graphs, thereby provide a method for the construction of CD-equienergetic graphs.

Theorem 2.1. *If G_1 and G_2 are any two graphs on n_1 and n_2 vertices respectively, then CD-polynomial of $G_1 \bigtriangledown G_2$ is*

$$\begin{aligned}\phi\Big(CD\big(G_1 \bigtriangledown G_2\big);\, x\Big) &= det\Big(xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\Big)\\ &\quad det\Big(xI_{n_1} - 2A(G_1) - A(\overline{G}_1) - 4J_{n_1\times n_2}\Big[xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\Big]^{-1} J_{n_2\times n_1}\Big)\end{aligned}$$

Proof. The general CD-matrix due to $G_1 \bigtriangledown G_2$ is

$$CD(G_1 \bigtriangledown G_2) = \begin{bmatrix} 2A(G_1) + A(\overline{G}_1) & 2J_{n_1\times n_2} \\ 2J_{n_2\times n_1} & 2A(G_2) + A(\overline{G}_2)\end{bmatrix}.$$

The CD-polynomial is

$$\begin{aligned}\phi\Big(CD\big(G_1 \bigtriangledown G_2\big);\, x\Big) &= det\Big(x\, I_{n_1+n_2} - CD(G_1 \bigtriangledown G_2)\Big)\\ &= det\left(\begin{array}{c|c} xI_{n_1} - 2A(G_1) - A(\overline{G}_1) & -2J_{n_1\times n_2} \\ \hline -2J_{n_2\times n_1} & xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\end{array}\right).\end{aligned}$$

Applying Proposition 1.1, result follows. □

Corollary 2.2. *If G_i are r_i-regular graphs on n_i vertices for $i = 1, 2$ then CD-polyomial of $G_1 \bigtriangledown G_2$ is*

$$\begin{aligned}\phi\Big(CD\big(G_1 \bigtriangledown G_2\big);\, x\Big) &= \prod_{i=2}^{n_1}\Big(x - \lambda_i(G_1) + 1\Big)\prod_{i=2}^{n_2}\Big(x - \lambda_i(G_2) + 1\Big)\\ &\quad \Big(\big(x - (n_1 + r_1 - 1)\big)\ \big(x - (n_2 + r_2 - 1)\big) - 4n_1n_2\Big).\end{aligned}$$

Proof. From Theorem 2.1, we have

$$\begin{aligned}\phi\Big(CD\big(G_1 \bigtriangledown G_2\big);\, x\Big) &= det\Big(xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\Big)\\ &\quad det\Big(xI_{n_1} - 2A(G_1) - A(\overline{G}_1) - 4J_{n_1\times n_2}\Big[xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\Big]^{-1} J_{n_2\times n_1}\Big)\end{aligned}$$

Since both G_1 and G_2 are regular, using the relation between the eigenvalues of graph and its complement, we have

$$\Big(xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\Big) = \Big(x - (n_2 + r_2 - 1)\Big)\prod_{i=2}^{n_2}\Big(x - \lambda_i(G_2) + 1\Big) \tag{2.1}$$

and

$$\begin{aligned}&det\Big(xI_{n_1} - 2A(G_1) - A(\overline{G}_1) - 4J_{n_1\times n_2}\Big[xI_{n_2} - 2A(G_2) - A(\overline{G}_2)\Big]^{-1} J_{n_2\times n_1}\Big)\\ &= \prod_{i=2}^{n_1}\Big(x - \lambda_i(G_1) + 1\Big)\frac{\Big(\big(x - (n_1 + r_1 - 1)\big)\big(x - (n_2 + r_2 - 1)\big) - 4n_1n_2\Big)}{\Big(x - (n_2 + r_2 - 1)\Big)}\end{aligned} \tag{2.2}$$

Since, $J_{n_1\times n_2}\Big[xI_{n_2}-2A(G_2)-A(\overline{G}_2)\Big]^{-1}J_{n_2\times n_1} = \dfrac{n_1\ n_2}{x-(n_2+r_2-1)}$ corresponding to the eigenvalue r_1 and zero otherwise. Hence, from equations (2.1) and (2.2), result follows. □

Remark 2.3. Let G_i be r_i-regular graphs on n_i vertices for $i = 1, 2$ with $\mu_{max}(G_1 \bigtriangledown G_2)$ as the largest CD-eigenvalue of $G_1 \bigtriangledown G_2$. If

(1) $n_1 = n_2$ and $r_1 = r_2$ then $\mu_{max}(G_1 \bigtriangledown G_2) = 3n_1 + r_1 - 1$
(2) $n_1 = n_2$ and $r_1 > r_2$ then $3n_1 + r_2 - 1 < \mu_{max}(G_1 \bigtriangledown G_2) < 3n_1 + r_1 - 1$
(3) $n_1 = n_2$ and $r_1 < r_2$ then $3n_1 + r_1 - 1 < \mu_{max}(G_1 \bigtriangledown G_2) < 3n_1 + r_2 - 1$
(4) $r_1 = r_2$ and $n_1 > n_2$ then
$2n_2 + n_1 + r_1 - 1 < \mu_{max}(G_1 \bigtriangledown G_2) < 2n_1 + n_2 + r_1 - 1$
(5) $r_1 = r_2$ and $n_1 < n_2$ then
$2n_1 + n_2 + r_1 - 1 < \mu_{max}(G_1 \bigtriangledown G_2) < 2n_2 + n_1 + r_1 - 1.$

Theorem 2.4. *If G_1 is a non-regular graph on n_1 vertices and G_2 is an r_2-regular graph on n_2 vertices then CD-polynomial of $G_1 \bigtriangledown G_2$ is*

$$\phi\Big(CD(G_1 \bigtriangledown G_2);\ x\Big) \;=\; \prod_{i=2}^{n_2}\Big(x-\lambda_i(G_2)+1\Big)\ \phi(Q;\ x)$$

where, $Q = \begin{bmatrix} 2A(G_1)+A(\overline{G}_1) & 2n_2J_{n_1\times 1} \\ 2J_{1\times n_1} & n_2+r_2-1 \end{bmatrix}$

Proof. As the matrix associated with $CD\big(G_1 \bigtriangledown G_2\big)$ is

$$CD(G_1 \bigtriangledown G_2) = \begin{bmatrix} 2A(G_1)+A(\overline{G}_1) & 2J_{n_1\times n_2} \\ 2J_{n_2\times n_1} & 2A(G_2)+A(\overline{G}_2) \end{bmatrix}.$$

Since G_2 is an r_2-regular graph on n_2 vertices, the block $2A(G_2) + A(\overline{G}_2)$ has the constant row sum of $(n_2 + r_2 - 1)$, taking all n_2 vertices in a single partition, above matrix is reduced to

$$Q = \begin{bmatrix} 2A(G_1)+A(\overline{G}_1) & 2n_2J_{n_1\times 1} \\ 2J_{1\times n_1} & n_2+r_2-1 \end{bmatrix},$$

which is called as quotient matrix. The eigenvalues that do not belong to the quotient due to the block $2A(G_2) + A(\overline{G}_2)$ coincide with eigenvalues associated to it except $n_2 + r_2 - 1$, is obtained due to equation (2.1) in the form of product of $(n_2 - 1)$ factors as:

$$\prod_{i=2}^{n_2}\Big(x-\lambda_i(G_2)+1\Big) \tag{2.3}$$

Hence, result follows from equation (2.3) and the polynomial due to Q. □

Remark 2.5. If G_1 is a non-regular graph on n_1 vertices and G_2 is an r_2-regular graph on n_2 vertices with $\mu_{max}(G_1 \bigtriangledown G_2)$ as the largest CD-eigenvalue of $G_1 \bigtriangledown G_2$ then

$$2n_1 + n_2 + r_2 - 1 < \mu_{max}(G_1 \bigtriangledown G_2) < 2(n_1 + n_2 - 1).$$

Proof. Proof directly follows from row sums of quotinet matrix Q by taking the maximum regularity in concern to the block $2A(G_1)+A(\overline{G}_1)$ which adds $2(n_1-1)$ to the sum $2(n_1+n_2-1)$. □

Corollary 2.6. *If G_1 is a non-regular graph on n_1 vertices, G_a and G_b be a pair of CD-equienergetic graphs on n_2 vertices, both of them are of r_2-regular. Then*

$$G_1 \bigtriangledown G_a \quad \text{and} \quad G_1 \bigtriangledown G_b$$

are also CD-equienergetic graphs.

Proof. As G_a and G_b are both r_2-regular graphs on n_2 vertices and are CD-equienergtic. Result follows from Theorem 2.4. □

Example 2.7. Consider path on 4 vertices (P_4), a pair of CD-equienergetic graphs G_a and G_b. Then graphs $P_4 \bigtriangledown G_a$ and $P_4 \bigtriangledown G_b$ are CD-equienergetic (Figure 1).
Since, $\mathcal{E}_{CD}\Big(P_4 \bigtriangledown G_a\Big) = \mathcal{E}_{CD}\Big(P_4 \bigtriangledown G_b\Big) = 32.41766915571303$.

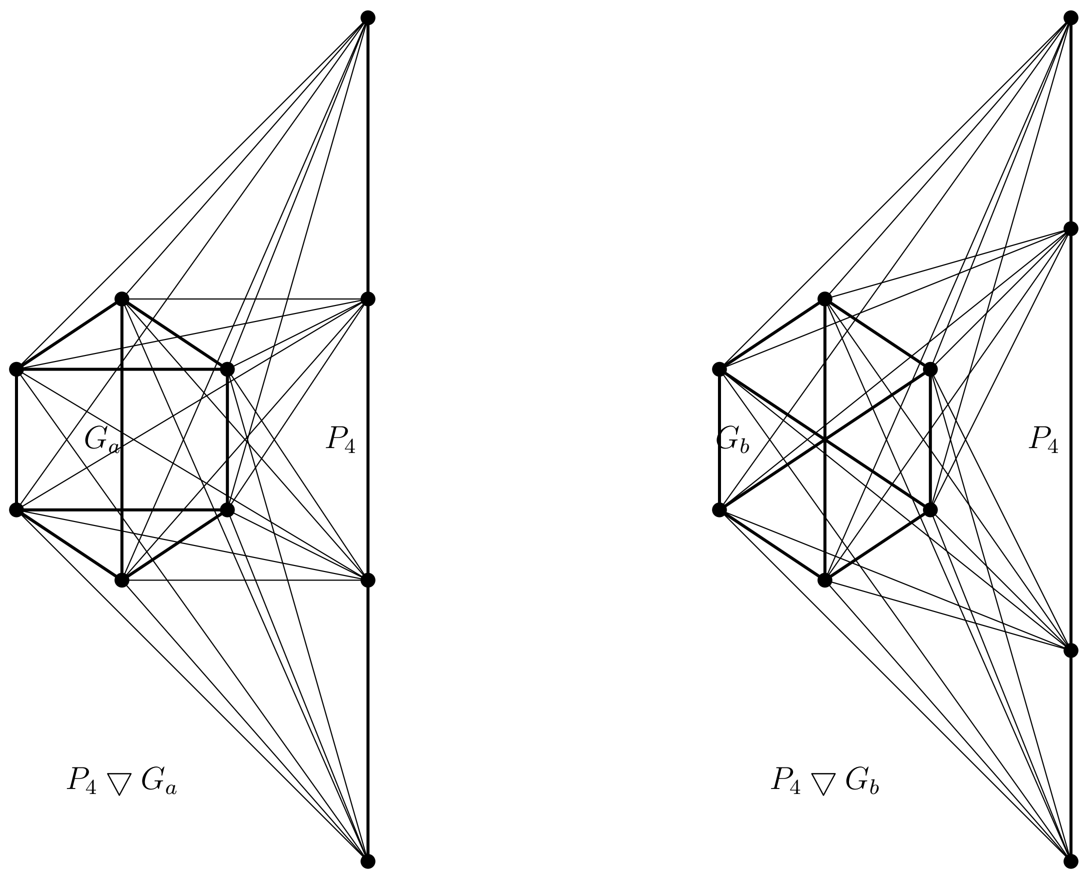

FIGURE 1.

Remark 2.8. For the structure $G_1 \bigtriangledown G_2$,

i. Corollary 2.2 signifies the study of CD-eigenvalues only with adjacency eigenvalues of both the graphs.

ii. Corollary 2.2 generalizes the results due to Ramane et al. in [7], where only regular graphs of diameter less than or equal to 2 are taken into account.

ii. Corollary 2.6 gives the construction of CD-equienergetic graphs by joining a regular graph (which is among a pair of CD-equienergetic graphs of same order and degree) with a non-regular graph.

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