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INVERSE NODAL PROBLEM IN MULTIPLICATIVE CASE

EMRE GÜNEŞ[1] AND EMRAH YILMAZ[2]

Abstract. In this study, the concepts of absolute value, limit, continuity, derivative and integral in multiplicative analysis are expressed. In addition, solution methods are expressed by considering differential equations in multiplicative calculus. Sturm-Liouville equation was established and the eigenvalues of the problem containing this equation and the spectral properties of the eigenfunctions were examined in multiplicative calculus. Inverse nodal problems, one of the inverse problem types in spectral analysis, are handled in multiplicative calculus and the solution of inverse nodal problem for the Sturm-Liouville equation is obtained in multiplicative calculus.

1. INTRODUCTION

The classical analysis that is most commonly used today was constructed by Newton and Leibnitz in the second half of the 17th century. Since the basic process in this analysis is aggregation, additive analysis has been called classical or Newtonian analysis. As a result of the ideas of establishing new analysis with different arithmetic operations based on classical analysis, many new types of analysis have emerged [1]. An example of this analysis is multiplicative calculus (or geometric analysis). This type of analysis are generally called non-Newtonian analysis in literature.

In the period from 1967 to 1970, Grossman and Katz introduced a new kind of definitions of derivatives and integrals, moving the roles of subtraction and addition operations to division and multiplication operations [2]. Later, Grossman and Katz published a book in 1972 [3] to describe general theory for non-Newtonian analysis which includes eight different kinds analysis. In addition to this study, Grossman examined exponential analysis in 1979 [4]. Numerous studies have been carried out in the following years related to this subject.

In 1980, Maginnis [5] introduced a study for probability theory as the first application of non-Newtonian analysis. In this study, non-Newtonian analysis was used as a statistical approach to human behavior and decision-making mechanisms. Grossman examined the derivative and integral in meta-analysis in 1981. In his book published in 1983, Grossman studied on some applications of bigeometric analysis [6,7]. In 1989, Hohlfeld et al. [8] used non-Newtonian analysis in their theories on atmospheric temperature. Stanley discussed some applications of multiplicative

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* Corresponding author.

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calculus in 1999. In this study, geometric analysis is called "multiplicative calculus" and some theorems about multiplicative derivative and multiplicative integral are given [9]. Campbell explained multiplicative Euler method in 1999 [10]. In 2001, Rybaczuk et al. [11] demonstrated that multiplicative calculus could be applied alternatively at stages when classical analysis was not sufficient. Cordova-Lepe dealt with the theory of elasticity, which had an important place in the economy in 2006, with the bigeometric derivative [12]. In 2007, Aniszewska presented the numerical multiplicative algorithm of Runge–Kutta to solve multiplicative differential equations [13].

Another important study in which the basic concepts of multiplicative calculus are defined was presented by Bashirov et al in 2008 [2]. This study was carried out in order to reinforce the foundations of the multiplicative calculus put forward by Grosmann and Katz and to reveal its benefits. In the study conducted by Rıza et al. in 2009, finite difference method for numerical solution of multiplicative differential equations and multiplicative Volterra differential equations was presented and applications were given [14]. Mısırlı and Özyapıcı developed multiplicative calculus in 2009 and gave a new definition of derivatives [15]. In 2010, Uzer examined complex analysis on multiplicative calculus. He pointed out that a multiplicative calculation of real-valued functions could be extended to a multiplicative type analysis that deals with complex-valued functions [16]. Bashirov and Riza examined the properties of complex multiplicative derivative for complex valued functions in 2011. They used Cauchy Riemann equations and put forward different complex multiplicative calculations [17]. Mısırlı and Gürefe used multiplicative calculus in many real life applications in 2011. They also gave a multiplicative version of Adams Bashforth–Moulton algorithms for numerical solution of multiplicative differential equations [18]. Bashirov et al. [19] used multiplicative calculus in 2012, stating that it would allow easier modeling for various problems in different fields. In 2012, Florack and Van Assen used multiplicative calculus in biomedical image analysis [20]. In 2013, Bashirov examined the concepts of curvilinear and double-fold integral in multiplicative case [21].

In 2014, Kadak and Efe examined Cauchy-Schwarz and triangular inequalities, using multiplicative norm properties of complex numbers [22]. Filip and Piatecki gave information in 2014 on how to apply the neoclassical model of external growth with non-Newtonian analysis [23]. In another study, they again used non-Newtonian analysis in the field of economics in the some year [24]. In 2015, Yener and Emiroğlu discussed the $q-$ analogy of some basic concepts of multiplicative calculus and called it $q-$ multiplicative calculus. In this study, $q-$ multiplicative calculus is introduced and some basic theorems related to derivative, integral and finite concepts are given [25].

In 2016, Yalçın et al. gave some basic definitions and characteristics of multiplicative Laplace transform based on the definitions and properties of usual Laplace transform [26]. They also obtained the solutions of some multiplicative differential equations with the help of Laplace transform. Boruah and Hazarika were informed about applications of bigeometric calculus in different branches of mathematics and economics in 2016 [27].

Yazıcı and Selvitop discussed some multiplicative methods for numerical solutions of partial differential equations in 2017. They concluded that these methods are more effective and useful than numerical methods [28]. In 2018, Waseem et al.

developed a technique within the framework of multiplicative calculus and applied this technique to the solution of population growth model and minimization problems [29]. Recently, multiplicative analysis has been applied to different topics in different fields [30–32].

Now let's inform about spectral theory and inverse problem that we'll study on multiplicative calculus. In spectral theory of linear differential operators, various properties of the direct problem and inverse problem have been investigated in detail by many authors in classical case. The direct problem here is the problem of determining the eigenvalues of differential operators and the corresponding eigenfunctions; the inverse problem is defined as the problem of finding the coefficient functions of the equation using spectral characteristics such as eigenvalue, eigenfunction, scattering data, normative constants, zeros of eigenvalues. Inverse problems of spectral analysis, has an important role in solving problems that arise in mathematics, physics, mechanics, electronics, geophysics, meteorology, seismology, medicine and other natural sciences [33].

The most important contribution to the development considered of inverse problems for Sturm-Liouville operator was a study formulated and considered by Ambarzumyan in 1929 [34]. In this study, Ambarzumyan showed that a single set of eigenvalues is sufficient to determine its potential function. However, contrary to this study, it is understood that a set of eigenvalues is not sufficient to determine the form of the operator. Furthermore, in the last 30-40 years, the theory of inverse nodal problems, a new class of inverse problems, has attracted the attention of the authors quite a lot.

Inverse nodal problem was first addressed by McLaughlin in 1988 [35]. For Sturm-Liouville problem, it has been proved that only knowledge of nodal points (zeros of eigen functions) is sufficient to determine the potential function [35, 36]. Later, some remarkable results were obtained by some authors on this topic. For example, some authors have reconstructed the potential function and its derivatives with the help of nodal datas in their study for different operators [37–39]. In addition, some authors have addressed the inverse nodal problem when boundary conditions depend on eigenparameter [40–43].

Inverse nodal problems for operators different from the classical Sturm-Liouville operator in that the Sturm-Liouville operator covers more than one potential function have been studied by some authors [44].

In this study, the inverse nodal problem for Sturm-Liouville equation will be constructed and solved in multiplicative calculus.

2. MAIN RESULTS

Here, we expres some basic notions and theorems related to our aim. Then, we express Sturm-Liouville equation on multiplicative calculus. Furthermore, we construct inverse nodal problem for Sturm-Liouville equation on this new calculus then solve it.

Definition 2.1. (Zero of a Function in the Classical Case) The zeros of a real, complex, or more generally vector-valued function are the points in the function's set of definitions where the function takes the value 0.

Example 2.2. Let $f:\mathbb{Z}\rightarrow\mathbb{Z}$, $f(x)=x^2-1$ be a function. The zeros of $f(x)$ are as follows in classical case. By definition 2.1, we get

$$x^2-1=0 \ \Rightarrow x=-1 \text{ and } x=1\ . \qquad (-1,1\in\mathbb{Z})$$

Thus, $x = -1$ and $x = 1$ are zeros of $f(x)$ in classical case.
Definition 2.3. (Zero of a Function in the Multiplicative Case) The zeros of a function in multiplicative calculus are the points in domain of function from which it takes the value 1.
Example 2.4. Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2 - 3$ be a function. The zeros of $f(x)$ are as follows in multiplicative case. By definition 2.3, we get

$$x^2 - 3 = 1 \Rightarrow x^2 = 4 \Rightarrow x = -2 \text{ and } x = 2 , \qquad (-2, 2 \in \mathbb{Z})$$

as the multiplicative zeros of $f(x)$.

For multiplicative Sturm-Liouville equation with similar logic, the nodal points of the problem will be obtained and solved by establishing the inverse nodal problem.
Definition 2.5. ([45]). The space $L_2^*[a, b]$ is defined by

$$L_2^*[a, b] = \left\{ f : \int_a^b [f(x) \odot f(x)]^{dx} < \infty \right\},$$

where f is a positive function.
Definition 2.6 ([45]). Let $q \in L_2^*[0, \pi]$ and λ be spectral parameter. The multiplicative Sturm-Liouville equation is defined by

$$(y^{**}(x))^{-1} y(x)^{q(x)} = y(x)^{\lambda}, \qquad x \in [0, \pi] \tag{1.1}$$

with the conditions

$$\begin{aligned} u(0, \lambda) &= e, \\ u_x^*(0, \lambda) &= e^h, \end{aligned} \tag{1.2}$$

$$\begin{aligned} v(0, \lambda) &= 1, \\ v_x^*(0, \lambda) &= e, \end{aligned} \tag{1.3}$$

respectively. The solutions of the problem (1.1),(1.2) and (1.1),(1.3) are

$$u(x, \lambda) = e^{cos\mu x + (h/\mu) sin\mu x} \left[\int_0^x u(t, \lambda)^{q(t) sin[\mu(x-t)]} \ dt \right]^{1/\mu}, \tag{1.4}$$

$$v(x, \lambda) = e^{(1/\mu) sin\mu x} \left[\int_0^x v(t, \lambda)^{q(t) sin[\mu(x-t)]} \ dt \right]^{1/\mu}, \tag{1.5}$$

respectively. The inverse nodal problem for these multiplicative Sturm-Liouville problems is the problem of obtaining its potential function $q(x)$ by using nodal multiplicative points. Using these eigenfunctions and the definition of a function in the multiplicative sense, the nodal points and nodal lengths of (1.1),(1.2) and (1.1),(1.3) multiplicative Sturm-Liouville problems will be obtained.

Let $0 < x_{1,*}^{(n)} < \ldots < x_{n-1,*}^{(n)} < \pi$ be the sequence of multiplicative nodal points of the problem (1.1),(1.2) and $x_{k,*}^{(n)}$ be the set of all nodal points of the given problem. Let $l_{k,*}^{(n)} = x_{k+1,*}^{(n)} - x_{k,*}^{(n)}$ be nodal length of corresponding problem in multiplicative case.

Theorem 2.7. The multiplicative nodal points and multiplicative nodal lengths of the problem (1.1),(1.2) are as follows.

$$x_{i,*}^{(n)}=\left(i-\frac{1}{2}\right)\frac{\pi}{\mu_n}+\frac{h}{\mu_n^2}+\frac{1}{2\mu_n^2}\int_0^{x_{i,*}^{(n)}}\left(1+cos\left(2\mu_n y\right)\right)q\left(y\right)dy+o\left(\frac{1}{\mu_n^3}\right),$$

$$l_{i,*}^{(n)}=\frac{\pi}{\mu_n}+\frac{1}{2\mu_n^2}\int_{x_i^{(n)}}^{x_{i+1,*}^{(n)}}\left(1+cos\left(2\mu_n y\right)\right)q\left(y\right)dy+o\left(\frac{1}{\mu_n^3}\right).$$

Proof: The eigenfunction of this problem is

$$u\left(x,\lambda\right)=e^{cos\mu x+(h/\mu)sin\mu x}\left[\int_0^x u\left(t,\lambda\right)^{q(t)\sin[(\mathrm{x}-\mathrm{t})]dt}\right]^{1/\mu}.$$

If we apply ln to both sides of above equality and the relationship between multiplicative analysis and classical analysis is used, we get

$$lnu\left(x,\lambda\right)=cos\mu x+\frac{h}{\mu}sin\mu x+\frac{1}{\mu}\int_0^x q\left(t\right)sin\left[\mu\left(x-t\right)\right]lnu\left(t,\lambda\right)dt,$$

and

$$U\left(x,\lambda\right)=cos\mu x+\frac{h}{\mu}sin\mu x+\frac{1}{\mu}\int_0^x q\left(t\right)sin\left[\mu\left(x-t\right)\right]U\left(t,\lambda\right)dt. \tag{1.6}$$

Let's rewrite the initial conditions according to $U\left(x,\lambda\right)$ by

$$u\left(0,\lambda\right)=e\ \Rightarrow lnu\left(0,\lambda\right)=lne\Rightarrow U\left(0,\lambda\right)=1,$$

$$u^*\left(0,\lambda\right)=e^h\ \Rightarrow \mathrm{e}^{\frac{u'}{u}(0,\lambda)}=e^h\Rightarrow \mathrm{e}^{\frac{u'}{u}(0,\lambda)}=h\Rightarrow U'\left(0,\lambda\right)=h.$$

With some iterations and trigonometric calculations, the following equation is obtained;

$$\begin{aligned}U\left(x,\lambda\right)&=cos(\mu x)+\frac{h}{\mu}sin(\mu x)+\frac{sin(\mu x)}{2\mu}\int_0^x q+\frac{1}{2\mu}\int_0^x sin\left[\mu\left(x-2y\right)\right]q\left(y\right)dy\\&-\frac{hcos(\mu x)}{2\mu^2}\int_0^x q\left(y\right)dy-\frac{cos(\mu x)}{4\mu^2}\int_0^x\int_0^x q\left(y\right)q\left(t\right)dtdy+o\left(\frac{1}{\mu^2}\right)\\&=cos(\mu x)+\frac{h}{\mu}sin(\mu x)+\frac{sin(\mu x)}{2\mu}\int_0^x\left(1+cos\left(2\mu y\right)\right)q\left(y\right)dy\\&-\frac{1}{2\mu}cos(\mu x)\int_0^x sin\left(2\mu y\right)q\left(y\right)dy-\frac{cos(\mu x)}{2\mu^2}\int_0^x q\left(y\right)dy\\&-\frac{1}{4\mu^2}cos(\mu x)\int_0^x\int_0^y q\left(y\right)q\left(t\right)dtdy+o\left(\frac{1}{\mu^2}\right).\end{aligned}$$

Here, $u\left(x,\lambda\right)=1\Rightarrow lnu\left(x,\lambda\right)=U\left(x,\lambda\right)=0$. Therefore, finding the multiplicative zeros of the function $u\left(x,\lambda\right)$ means finding the classical zeros of the function $U\left(x,\lambda\right)$. If $U\left(x,\lambda\right)=0$ where $sin\left(\mu x\right)\neq 0$, then

$$\begin{aligned}cot\left(\mu x\right)=&-\frac{h}{\mu}-\frac{1}{2\mu}\int_0^x\left(1+cos\left(2\mu y\right)\right)q\left(y\right)dy+\frac{cos(\mu x)}{2\mu}\int_0^x sin\left(2\mu y\right)q\left(y\right)dy\\&+\frac{h}{2\mu^2}cot(\mu x)\int_0^x q+\frac{cot(\mu x)}{4\mu^2}\int_0^x\int_0^x q\left(y\right)q\left(t\right)dtdy+o\left(\frac{1}{\mu^2}\right)\end{aligned}$$

is obtained. From here, we get

$$cot(\mu x)\left(1+o\left(\frac{1}{\mu}\right)\right) = -\frac{h}{\mu}-\frac{1}{2\mu}\int_0^x (1+cos\,(\mu y))\,q\,(y)\,dy+o\left(\frac{1}{\mu^2}\right)$$

and

$$cot(\mu x) = -\frac{h}{\mu}-\frac{1}{2\mu}\int_0^x (1+cos\,(\mu y))\,q\,(y)\,dy+o\left(\frac{1}{\mu^2}\right).$$

If $\mu=\mu_n$, we get $\left|\mu_n x_{i,*}^{(n)}-\left(i-\frac{1}{2}\right)\pi\right|<\frac{\pi}{4}$ as $x=x_{i,*}^{(n)}$. From the Taylor's series expansion of the cotangent function, we get

$$\mu_n x_{i,*}^{(n)} = \left(i-\frac{1}{2}\right)\pi+\frac{h}{\mu_n}+\frac{1}{2\mu_n}\int_0^{x_{i,*}^{(n)}} (1+cos\,(2\mu_n y))\,q\,(y)\,dy+o\left(\frac{1}{\mu_n^2}\right).$$

Thus,

$$x_{i,*}^{(n)} = \left(i-\frac{1}{2}\right)\frac{\pi}{\mu_n}+\frac{h}{\mu_n^2}+\frac{1}{2\mu_n^2}\int_0^{x_{i,*}^{(n)}} (1+cos\,(2\mu_n y))\,q\,(y)\,dy+o\left(\frac{1}{\mu_n^3}\right),$$

is obtained. By the definition of nodal length, we get

$$l_{i,*}^{(n)} = \frac{\pi}{\mu_n}+\frac{1}{2\mu_n^2}\int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} (1+cos\,(2\mu_n y))\,q\,(y)\,dy+o\left(\frac{1}{\mu_n^3}\right),$$

and

$$U\left(x_{i,*}^{(n)},\lambda\right)=0\Rightarrow u\left(x_{i,*}^{(n)},\lambda\right)=1.$$

This indicates that $x_{i,*}^{(n)}$ are multiplicative nodal points of $u\,(x,\lambda)$. Indeed, since $U\left(x_{i,*}^{(n)},\lambda\right)=0$, we get $lnu\left(x_{i,*}^{(n)},\lambda\right)=0\Rightarrow u\left(x_{i,*}^{(n)},\lambda\right)=1$. This shows that $x_{i,*}^{(n)}$ are multiplicative nodal points of the considered multiplicative Sturm-Liouville problem.

Theorem 2.8. The multiplicative nodal points and multiplicative nodal lengths of the problem (1.1),(1.3) have following expressions.

$$x_{i,*}^{(n)} = \frac{i\pi}{\mu_n}+\frac{1}{2\mu_n^2}\int_0^{x_{i,*}^{(n)}} q\,(y)\,dy+o\left(\frac{1}{\mu_n^2}\right),$$

$$l_{i,*}^{(n)} = \frac{\pi}{\mu_n}+\int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q\,(y)\,dy+o\left(\frac{1}{\mu_n^2}\right).$$

Proof: The eigenfunction of (1.1),(1.3) is

$$v\,(x,\lambda)=e^{(1/\mu)sin\mu x}\left[\int_0^x v\,(t,\lambda)^{q(t)sin[\mu(x-t)]}\;dt\right]^{1/\mu}.$$

By taking ln of both sides of above equality and considering relationship between multiplicative calculus and classical analysis, we get

$$lnv\,(x,\lambda)=\frac{1}{\mu}sin\mu x+\frac{1}{\mu}\int_0^x q\,(t)\,sin\,[\mu\,(x-t)]\,lnv\,(t,\lambda)dt\;,$$

$$V\,(x,\lambda)=\frac{1}{\mu}sin\mu x+\frac{1}{\mu}\int_0^x q\,(t)\,sin\,[\mu\,(x-t)]\,V\,(t,\lambda)\,dt. \tag{1.7}$$

Let's rewrite the initial conditions written according to $V(x,\lambda)$.

$$v(0,\lambda)=1 \ \Rightarrow lnv(0,\lambda)=ln1\Rightarrow V(x,\lambda)=0,$$

$$v_x^*(0,\lambda)=e \ \Rightarrow \mathrm{e}^{\frac{v'}{v}(0,\lambda)}=e\Rightarrow \frac{v'}{v}(0,\lambda)=1\Rightarrow V'(0,\lambda)=1.$$

By some trigonometric calculations, following equation is obtained

$$V(x,\lambda)=\frac{1}{\mu}sin(\mu x)\ -\frac{1}{2\mu^2}cos(\mu x)\ \int_0^x q(y)\,dy+o\left(\frac{1}{\mu^2}\right).$$

Here, $v(x,\lambda)=1\Rightarrow lnv(x,\lambda)=V(x,\lambda)=0$. Therefore, finding the multiplicative zeros of the function $v(x,\lambda)$ means finding the classical zeros of the function $V(x,\lambda)$. If $V(x,\lambda)=0$ where $sin(\mu x)\neq 0$, then

$$\begin{aligned} sin(\mu x)\ &=\frac{1}{2\mu}cos(\mu x)\ \int_0^x q(y)\,dy+o\left(\frac{1}{\mu}\right)\ ,\\ tan(\mu x)\ &=\frac{1}{2\mu}\int_0^x q(y)\,dy+o\left(\frac{1}{\mu}\right),\end{aligned}$$

is obtained. By setting $\mu=\mu_n$ from the $x=x_{i,*}^{(n)}$ Taylor's series expansion of tangent function gives

$$\mu_n x_{i,*}^{(n)}=i\pi+\frac{1}{2\mu}\int_0^{x_{i,*}^{(n)}} q(y)\,dy+o\left(\frac{1}{\mu}\right).$$

Thus,

$$x_{i,*}^{(n)}=\frac{i\pi}{\mu_n}+\frac{1}{2\mu_n^2}\int_0^{x_{i,*}^{(n)}} q(y)\,dy+o\left(\frac{1}{\mu^2}\right),$$

is obtained. From the definition $l_{i,*}^{(n)}=x_{i+1,*}^{(n)}-x_{i,*}^{(n)}$, we get

$$l_{i,*}^{(n)}=\frac{(i+1)\pi}{\mu_n}+\frac{1}{2\mu_n^2}\int_0^{x_{i+1,*}^{(n)}} q(y)\,dy+o\left(\frac{1}{\mu^2}\right)\ -\frac{i\pi}{\mu_n}-\frac{1}{2\mu_n^2}\int_0^{x_{i,*}^{(n)}} q(y)\,dy+o\left(\frac{1}{\mu^2}\right)$$

$$\Rightarrow l_{i,*}^{(n)}=\frac{\pi}{\mu_n}+\int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy+o\left(\frac{1}{\mu^2}\right),$$

and

$$lnv\left(x_{i,*}^{(n)},\lambda\right)=0\Rightarrow v\left(x_{i,*}^{(n)},\lambda\right)=1.$$

This gives that $x_{i,*}^{(n)}$ are multiplicative nodal points of $v(x,\lambda)$. Also $l_{i,*}^{(n)}$ is the multiplicative nodal length $v(x,\lambda)$.

Lemma 2.9. Let $f\in L_*^1(0,\pi)$. Then,

$$\lim_{n\to\infty}\left[\int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} f(t)^{dt}\right]^{\frac{\mu_n}{\pi}}=f(x),$$

for almost $x\in(0,\pi)$.

Theorem 2.10. The potential function $q\in L_*^1(0,\pi)$ of the problem (1.1),(1.2) satisfies

$$q(x)=\lim_{n\to\infty}\left(\frac{\mu_n}{\pi}l_{i,*}^{(n)}-1\right),$$

where $x\in(0,\pi)$.

Proof: Indeed,

$$l_{i,*}^{(n)} = \frac{\pi}{\mu_n} + \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy + o\left(\frac{1}{\mu^2}\right) \ ,$$

$$\frac{\mu_n}{\pi} l_{i,*}^{(n)} = 1 + \frac{\mu_n}{\pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy + o\left(\frac{1}{\mu_n}\right) \ ,$$

$$\frac{\mu_n l_{i,*}^{(n)}}{\pi} - 1 = \frac{\mu_n}{\pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy + o\left(\frac{1}{\mu_n}\right) \ ,$$

$$q(x) = \lim_{n\to\infty} \left(\frac{\mu_n}{\pi} l_{i,*}^{(n)} - 1\right)$$

is the potential function of the given problem in multiplicative case.

Theorem 2.11. The potential function $q \in L_*^1(0,\pi)$ of the problem (1.1),(1.3) satisfies

$$q(x) = \lim_{n\to\infty} 2\mu_n^2 \left(\frac{\mu_n l_{i,*}^{(n)}}{\pi} - 1\right),$$

on $(0,\pi)$.

Proof: Indeed,

$$l_{i,*}^{(n)} = \frac{\pi}{\mu_n} + \frac{1}{2\mu_n^2} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} (1 + cos(2\mu_n y))\, q(y)\,dy + o\left(\frac{1}{\mu_n^3}\right) \ ,$$

$$\frac{\mu_n}{\pi} l_{i,*}^{(n)} = 1 + \frac{1}{2\mu_n \pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} (1 + cos(2\mu_n y))\, q(y)\,dy + o\left(\frac{1}{\mu_n^2}\right) \ ,$$

$$\frac{\mu_n}{\pi} l_{i,*}^{(n)} - 1 = \frac{1}{2\mu_n \pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy + \frac{1}{2\mu_n \pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} cos(2\mu_n y)\, q(y)\,dy + o\left(\frac{1}{\mu_n^2}\right)$$

is obtained. If we multiply both sides by $2\mu_n^2$, we get

$$2\mu_n^2 \left(\frac{\mu_n}{\pi} l_{i,*}^{(n)} - 1\right) = \frac{\mu_n}{\pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy + \frac{\mu_n}{\pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} cos(2\mu_n y)\, q(y)\,dy.$$

By taking limit of both sides,

$$\lim_{n\to\infty} 2\mu_n^2 \left(\frac{\mu_n}{\pi} l_{i,*}^{(n)} - 1\right) = \lim_{n\to\infty} \frac{\mu_n}{\pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} q(y)\,dy + \lim_{n\to\infty} \frac{\mu_n}{\pi} \int_{x_{i,*}^{(n)}}^{x_{i+1,*}^{(n)}} cos(2\mu_n y)\, q(y)\,dy \ ,$$

$$q(x) = \lim_{n\to\infty} 2\mu_n^2 \left(\frac{\mu_n l_{i,*}^{(n)}}{\pi} - 1\right),$$

is obtained. This is the potential function of the problem (1.1),(1.3).

REMARK: The nodal points and nodal lengths of below multiplicative Sturm-Liouville equation

$$\left(y^{**}\right)^{-1} y^{q(x)} = y^{\lambda},$$

are also nodal points and nodal lengths of the following corresponding equation

$$y'' y - \left(y'\right)^2 + \left[(\lambda - q(x))\, lny\right] y^2 = 0.$$

While it is difficult to find the nodal points and nodal lengths of the problem involving this nonlinear differential equation in classical analysis, it is easily obtained in multiplicative calculus.

3. CONCLUSIONS

In non-Newtonian analysis, the results to be obtained are of great importance because the derivative, integral, structure of the equation and solution methods are different. In this study, the inverse nodal problem was defined for the first time in multiplicative (or geometric) calculus, which is a special case of the analysis produced with the help of arithmetic, one of the non-Newtonian analysis, and this problem was solved by using multiplicative calculus techniques. First, the concept of the zero of a function was expressed concretely in multiplicative calculus, obtaining the nodal points of the established multiplicative Sturm-Liouville equation and problem. Then, the nodal length of the problem was obtained by using definition. Finally, using some special theorems, the potential function in multiplicative Sturm-Liouville equation was obtained.

This study will provide an important motivation to mathematicians working on this subject that problems in spectral analysis can be addressed in non-Newtonian analysis. These are the obvious problems that the inverse nodal problem can be established for quadratic, harmonic, sigmoidal, parabolic analyzes from the analyzes produced with the help of arithmetics and that this problem can be solved under different boundary conditions for different operators. Operators such as Dirac, diffusion, string, Bessel can be defined in all these analysis produced with the help of arithmetic and direct and inverse problems can be solved for these operators.

Conflicts of Interest

The author declares no conflicts of interest. No competing financial interests exist.

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Emre GÜNEŞ
Department of Mathematics, Faculty of Science, Firat University, Elazig, Türkiye
Email address: emre4245@gmail.com

Emrah YILMAZ
Department of Mathematics, Faculty of Science, Firat University, Elazig, Türkiye
Email address: emrah231983@gmail.com