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COUPLED FIXED POINT RESULTS ON COMPLEX PARTIAL METRIC SPACES FOR CONTRACTIVE TYPE CONDITIONS

GURUCHARAN SINGH SALUJA

Abstract. In this work, we obtain some coupled fixed point results on complex partial metric spaces satisfying contractive type conditions. Furthermore, we provide some consequences of the demonstrated result. An example is given to support the result. The results of findings in this work extend and generalize several results from the existing literature. Specially, our results extend the corresponding results of *Aydi* [1].

1. Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool. In 1922, *Stefan Banach* ([3]) formulated the notion of contraction and proved the famous theorem, scientist and mathematicians around the world are publishing new results that are related either to establish a generalization of metric space or to get a improvement of contractive conditions.

There are many extensions of the famous Banach contraction principle, which states that every self mapping $\mathcal{T}$ defined on a complete metric space $(\mathcal{M}, \rho)$ satisfying

$$\rho(\mathcal{T}(\mu), \mathcal{T}(\nu)) \leq k\, \rho(\mu, \nu), \tag{1.1}$$

for all $\mu, \nu \in \mathcal{M}$, where $k \in (0,1)$, has a unique fixed point and for every $x_0 \in \mathcal{M}$, a sequence $\{\mathcal{T}^n x_0\}_{n\geq 1}$ is convergent to the fixed point. Inequality (1.1) also implies the continuity of $\mathcal{T}$.

The Banach contraction principle is very effectively utilized for solving various applied problems in mathematics and engineering. Many authors extended and generalized Banach fixed point theorem in different ways (see, e.g., [7], [9], [10], [11]).

In 2011, *Azam et al.* [2] introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The results proved by *Azam*

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* Corresponding author.
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et al. [2] and *Bhatt et al.* [5] via rational inequality in a complex valued metric space as a contractive condition. Complex valued metric space is very useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics, applied physics, mechanical engineering, thermodynamics and electrical engineering.

In 2017, *Dhivya and Marudai* [8] introduced new spaces called complex partial metric space and demonstrated the existence of common fixed point results under contractive condition involving rational expression.

On the other hand, *Bhashkar and Lakshmikantham* [4] introduced the concept of coupled fixed points in ordered spaces and applied their results to boundary value problems for the unique solution. *Ciric and Lakshmikantham* [6] introduced the concept of coupled coincidence, common fixed points to nonlinear contractions in ordered metric spaces. More results on coupled fixed points, coupled coincidence points and common coupled fixed points in various spaces, one can see [1, 12, 13, 14, 15, 16, 18, 19, 20, 21] and many others.

Aydi [1] demonstrated a number of coupled fixed point results for contractive type conditions in partial metric spaces and gave some examples in support of the results. Recently, *Kim et al.* [13] demonstrated some common coupled fixed point theorems for contractive type conditions in the setting of partial metric spaces.

Inspired and motivated by the works of *Aydi* [1], *Bhashkar and Lakshmikantham* [4], *Dhivya and Marudai* [8] and others, the purpose of this work is to investigate some coupled fixed point results for contractive type conditions in the framework of complex partial metric spaces. Also an example is given to support the result. The results of findings in this work extend and generalize several previously published findings from the existing literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $c_1, c_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$c_1 \precsim c_2$ if and only if $Re(c_1) \leq Re(c_2)$, $Im(c_1) \leq Im(c_2)$. It follows that $c_1 \precsim c_2$ if one of the following conditions is satisfied:

(i) $Re(c_1) = Re(c_2)$, $Im(c_1) < Im(c_2)$;
(ii) $Re(c_1) < Re(c_2)$, $Im(c_1) = Im(c_2)$;
(iii) $Re(c_1) < Re(c_2)$, $Im(c_1) < Im(c_2)$;
(iv) $Re(c_1) = Re(c_2)$, $Im(c_1) = Im(c_2)$.

In particular, we will write $c_1 \precnsim c_2$ if $c_1 \neq c_2$ and one of (i), (ii), and (iii) is satisfied and we will write $c_1 \prec c_2$ if only (iii) is satisfied. Notice that:

(a) If $0 \lesssim c_1 \precnsim c_2$, then $|c_1| < |c_2|$,
(b) If $c_1 \precsim c_2$ and $c_2 \prec c_3$, then $c_1 \prec c_3$,
(c) If $p, q \in \mathbb{R}$ and $p \leq q$, then $pz \precsim qz$ for all $z \in \mathbb{C}$.

In 2017, Dhivya and Marudai [8] define the following.

Definition 2.1. ([8]) Let $\mathcal{M}$ be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A complex partial metric space on $\mathcal{M}$ is a function $p_c\colon \mathcal{M}\times\mathcal{M}\to\mathbb{C}^+$ such that for all $\zeta,\eta,\theta\in\mathcal{M}$:

($\mathcal{CPM}$1) $0\precsim p_c(\zeta,\zeta)\precsim p_c(\zeta,\eta)$ (small self-distance),
($\mathcal{CPM}$2) $p_c(\zeta,\eta)=p_c(\eta,\zeta)$ (symmetry),
($\mathcal{CPM}$3) $p_c(\zeta,\zeta)=p_c(\zeta,\eta)=p_c(\eta,\eta)$ if and only if $\zeta=\eta$ (equality),
($\mathcal{CPM}$4) $p_c(\zeta,\eta)\precsim p_c(\zeta,\theta)+p_c(\theta,\eta)-p_c(\theta,\theta)$ (triangularity).

A complex partial metric space is a pair $(\mathcal{M},p_c)$ such that $\mathcal{M}$ is a non-empty set and p_c is complex partial metric on $\mathcal{M}$.

For the complex partial metric p_c on $\mathcal{M}$, the function $d_{p_c}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{C}^+$ given by

$$p_c^t(\zeta,\eta)=2p_c(\zeta,\eta)-p_c(\zeta,\zeta)-p_c(\eta,\eta), \tag{2.1}$$

is a (usual) metric on $\mathcal{M}$.

Also note that each complex partial metric p_c on $\mathcal{M}$ generates a T_0 topology τ_{p_c} on $\mathcal{M}$ with the base family of open p_c-balls $\{\mathcal{B}_{p_c}(\zeta,\varepsilon):\zeta\in\mathcal{M},\varepsilon>0\}$ where

$$\mathcal{B}_{p_c}(\zeta,\varepsilon)=\{\eta\in\mathcal{M}:p_c(\zeta,\eta)<p_c(\zeta,\zeta)+\varepsilon\},$$

for all $\zeta\in\mathcal{M}$ and $0\prec\varepsilon\in\mathbb{C}^+$.

Similarly, closed p_c-ball is defined as

$$\mathcal{B}_{p_c}[\zeta,\varepsilon]=\{\eta\in\mathcal{M}:p_c(\zeta,\eta)\le p_c(\zeta,\zeta)+\varepsilon\},$$

for all $\zeta\in\mathcal{M}$ and $0\prec\varepsilon\in\mathbb{C}^+$.

A complex valued metric space is a complex partial metric space. But a complex partial metric space need not be a complex valued metric space. The following example illustrates such a complex partial metric space.

Example 2.2. ([8]) Let $\mathcal{M}=[0,\infty)$ endowed with complex partial metric p_c is defined by $p_c\colon \mathcal{M}\times\mathcal{M}\to\mathbb{C}^+$ with

$$p_c(\zeta,\eta)=\max\{\zeta,\eta\}+i\max\{\zeta,\eta\}\ \text{for all}\ \zeta,\eta\in\mathcal{M}.$$

It is easy to verify that $(\mathcal{M},p_c)$ is a complex partial metric space and note that self distance need not be zero, for example, $p_c(1,1)=1+i\neq 0$. Now the metric induced by p_c is as follows

$$p_c^t(\zeta,\eta)=2p_c(\zeta,\eta)-p_c(\zeta,\zeta)-p_c(\eta,\eta),$$

without loss of generality suppose $\zeta\ge\eta$, then

$$p_c^t(\zeta,\eta)=2\big[\max\{\zeta,\eta\}+i\max\{\zeta,\eta\}\big]-(\zeta+i\zeta)-(\eta+i\eta).$$

Therefore,

$$p_c^t(\zeta,\eta)=|\zeta-\eta|+i|\zeta-\eta|.$$

Definition 2.3. ([8]) Let $(\mathcal{M},p_c)$ be a complex partial metric space ($CPMS$). A sequence $\{\zeta_n\}$ in a $CPMS$ $(\mathcal{M},p_c)$ converges to ζ if and only if for every $\varepsilon\in\mathbb{C}^+$ with $0\prec\varepsilon$, there exists $N_0\in\mathbb{N}$ such that for all $n\ge N_0$, we have $p_c(\zeta_n,\zeta)\prec\varepsilon$ or $\zeta_n\in\mathcal{B}_{p_c}(\zeta,\varepsilon)$ and we denote this by $\zeta_n\to\zeta$ or $\lim_{n\to\infty}\zeta_n=\zeta$.

Definition 2.4. ([8]) Let $(\mathcal{M}, p_c)$ be a complex partial metric space $(CPMS)$. A sequence $\{\zeta_n\}$ in a $CPMS$ $(\mathcal{M}, p_c)$ is called a Cauchy sequence if for every $\varepsilon \in \mathbb{C}^+$ with $0 \prec \varepsilon$ and $a \in \mathbb{C}^+$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, we have $|p_c(\zeta_n, \zeta_m) - a| < \varepsilon$.

Definition 2.5. ([8]) Let $(\mathcal{M}, p_c)$ be a complex partial metric space $(CPMS)$.

(1) A $CPMS$ $(\mathcal{M}, p_c)$ is said to be complete if a Cauchy sequence $\{\zeta_n\}$ in $\mathcal{M}$ converges with respect to τ_{p_c} to a point $\zeta \in \mathcal{M}$ such that $p_c(\zeta, \zeta) = \lim_{n,m\to\infty} p_c(\zeta_n, \zeta_m)$.

(2) A napping $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ is said to be continuous at $\zeta_0 \in \mathcal{M}$ if for every $0 \prec \varepsilon$, there exists $\delta > 0$ such that $\mathcal{F}\big(\mathcal{B}_{p_c}(\zeta_0, \delta)\big) \subset \mathcal{B}_{p_c}\big(\mathcal{F}(\zeta_0), \varepsilon\big)$.

Definition 2.6. Let $\mathcal{M}$ be a non-empty set and let $\mathcal{A}, \mathcal{B}\colon \mathcal{M} \to \mathcal{M}$ be two self mappings of $\mathcal{M}$. Then a point $t \in \mathcal{M}$ is called a

- fixed point of operator $\mathcal{A}$ if $\mathcal{A}(t) = t$,
- common fixed point of $\mathcal{A}$ and $\mathcal{B}$ if $\mathcal{A}(t) = \mathcal{B}(t) = t$.

Definition 2.7. Let $(\mathcal{M}, p_c)$ be a complex partial metric space $(CPMS)$. Then an element $(\zeta, \eta) \in \mathcal{M} \times \mathcal{M}$ is said to be a coupled fixed point of the mapping $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ if $F(\zeta, \eta) = \zeta$ and $F(\eta, \zeta) = \eta$.

Example 2.8. Let $\mathcal{M} = [0, +\infty)$ and $F\colon \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ defined by $F(\zeta, \eta) = \frac{\zeta+\eta}{3}$ for all $\zeta, \eta \in \mathcal{M}$. One can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.9. Let $\mathcal{M} = [0, +\infty)$ and $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be defined by $F(\zeta, \eta) = \frac{\zeta+\eta}{2}$ for all $\zeta, \eta \in \mathcal{M}$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Lemma 2.10. *([8]) Let $(\mathcal{M}, p_c)$ be a complex partial metric space $(CPMS)$. A sequence $\{\zeta_n\}$ is a Cauchy sequence in the $CPMS$ $(\mathcal{M}, p_c)$, then $\{\zeta_n\}$ is Cauchy in a metric space $(\mathcal{M}, p_c^t)$.*

3. Main results

In this section, we shall prove some unique coupled fixed point theorems for contractive type conditions in the framework of complex partial metric spaces.

Theorem 3.1. *Let $(\mathcal{M}, p_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfying the following contractive condition for all $\zeta, \eta, \theta, \lambda \in \mathcal{M}$:*

$$\begin{aligned} p_c(F(\zeta, \eta), F(\theta, \lambda)) \precsim\ & a_1\, p_c(\zeta, \theta) + a_2\, p_c(\eta, \lambda) + a_3\, p_c(F(\zeta, \eta), \zeta) \\ & + a_4\, p_c(F(\theta, \lambda), \theta) + a_5\, p_c(F(\zeta, \eta), \theta) \\ & + a_6\, p_c(F(\theta, \lambda), \zeta) + a_7\, p_c(F(\eta, \zeta), \lambda) \\ & + a_8\, p_c(F(\lambda, \theta), \eta), \end{aligned} \tag{3.1}$$

where $a_1, a_2, \ldots, a_8$ are nonnegative constants with $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 + a_7 + 2a_8 < 1$. Then F has a unique coupled fixed point.

Proof. Choose $\zeta_0, \eta_0 \in \mathcal{M}$. Set $\zeta_1 = F(\zeta_0, \eta_0)$ and $\eta_1 = F(\eta_0, \zeta_0)$. Repeating this process, we obtain two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\mathcal{M}$ such that $\zeta_{n+1} = F(\zeta_n, \eta_n)$

and $\eta_{n+1} = F(\eta_n, \zeta_n)$. Let $A_n = |p_c(\zeta_n, \zeta_{n+1})|$ and $B_n = |p_c(\eta_n, \eta_{n+1})|$. Then, from equations (3.1) and using ($\mathcal{CPM}$1), ($\mathcal{CPM}$2), ($\mathcal{CPM}$4), we have

$$\begin{aligned} p_c(\zeta_n, \zeta_{n+1}) &= p_c(F(\zeta_{n-1}, \eta_{n-1}), F(\zeta_n, \eta_n)) \\ &\precsim a_1\, p_c(\zeta_{n-1}, \zeta_n) + a_2\, p_c(\eta_{n-1}, \eta_n) \\ &\quad + a_3\, p_c(F(\zeta_{n-1}, \eta_{n-1}), \zeta_{n-1}) + a_4\, p_c(F(\zeta_n, \eta_n), \zeta_n) \\ &\quad + a_5\, p_c(F(\zeta_{n-1}, \eta_{n-1}), \zeta_n) + a_6\, p_c(F(\zeta_n, \eta_n), \zeta_{n-1}) \\ &\quad + a_7\, p_c(F(\eta_{n-1}, \zeta_{n-1}), \eta_n) + a_8\, p_c(F(\eta_n, \zeta_n), \eta_{n-1}) \\ &= a_1\, p_c(\zeta_{n-1}, \zeta_n) + a_2\, p_c(\eta_{n-1}, \eta_n) \\ &\quad + a_3\, p_c(\zeta_n, \zeta_{n-1}) + a_4\, p_c(\zeta_{n+1}, \zeta_n) \\ &\quad + a_5\, p_c(\zeta_n, \zeta_n) + a_6\, p_c(\zeta_{n+1}, \zeta_{n-1}) \\ &\quad + a_7\, p_c(\eta_n, \eta_n) + a_8\, p_c(\eta_{n+1}, \eta_{n-1}) \\ &\precsim a_1\, p_c(\zeta_{n-1}, \zeta_n) + a_2\, p_c(\eta_{n-1}, \eta_n) \\ &\quad + a_3\, p_c(\zeta_{n-1}, \zeta_n) + a_4\, p_c(\zeta_n, \zeta_{n+1}) \\ &\quad + a_5\, p_c(\zeta_n, \zeta_{n+1}) + a_6\, [p_c(\zeta_{n+1}, \zeta_n) + p_c(\zeta_n, \zeta_{n-1}) \\ &\quad - p_c(\zeta_n, \zeta_n)] + a_7\, p_c(\eta_n, \eta_{n+1}) + a_8\, [p_c(\eta_{n+1}, \eta_n) \\ &\quad + p_c(\eta_n, \eta_{n-1}) - p_c(\eta_n, \eta_n)] \\ &\precsim a_1\, p_c(\zeta_{n-1}, \zeta_n) + a_2\, p_c(\eta_{n-1}, \eta_n) \\ &\quad + a_3\, p_c(\zeta_{n-1}, \zeta_n) + a_4\, p_c(\zeta_n, \zeta_{n+1}) \\ &\quad + a_5\, p_c(\zeta_n, \zeta_{n+1}) + a_6\, [p_c(\zeta_n, \zeta_{n+1}) + p_c(\zeta_{n-1}, \zeta_n)] \\ &\quad + a_7\, p_c(\eta_n, \eta_{n+1}) + a_8\, [p_c(\eta_n, \eta_{n+1}) + p_c(\eta_{n-1}, \eta_n)] \\ &= (a_1 + a_3 + a_6)\, p_c(\zeta_{n-1}, \zeta_n) + (a_2 + a_8)\, p_c(\eta_{n-1}, \eta_n) \\ &\quad + (a_4 + a_5 + a_6)\, p_c(\zeta_n, \zeta_{n+1}) + (a_7 + a_8)\, p_c(\eta_n, \eta_{n+1}), \end{aligned}$$

which implies that

$$\begin{aligned} |p_c(\zeta_n, \zeta_{n+1})| &\leq (a_1 + a_3 + a_6)\, |p_c(\zeta_{n-1}, \zeta_n)| + (a_2 + a_8)\, |p_c(\eta_{n-1}, \eta_n)| \\ &\quad + (a_4 + a_5 + a_6)\, |p_c(\zeta_n, \zeta_{n+1})| \\ &\quad + (a_7 + a_8)\, |p_c(\eta_n, \eta_{n+1})|. \end{aligned} \tag{3.2}$$

Similarly, one can prove that

$$\begin{aligned} |p_c(\eta_n, \eta_{n+1})| &= |p_c(F(\eta_{n-1}, \zeta_{n-1}), F(\eta_n, \zeta_n))| \\ &\leq (a_1 + a_3 + a_6)\, |p_c(\eta_{n-1}, \eta_n)| + (a_2 + a_8)\, |p_c(\zeta_{n-1}, \zeta_n)| \\ &\quad + (a_4 + a_5 + a_6)\, |p_c(\eta_n, \eta_{n+1})| \\ &\quad + (a_7 + a_8)\, |p_c(\zeta_n, \zeta_{n+1})|. \end{aligned} \tag{3.3}$$

Hence from equations (3.2) and (3.3), we obtain

$$|p_c(\zeta_n, \zeta_{n+1})| + |p_c(\eta_n, \eta_{n+1})|$$

$$
\begin{aligned}
&\leq (a_1+a_3+a_6)\left[|p_c(\zeta_{n-1},\zeta_n)|+|p_c(\eta_{n-1},\eta_n)|\right]\\
&\quad +(a_2+a_8)\left[|p_c(\zeta_{n-1},\zeta_n)|+|p_c(\eta_{n-1},\eta_n)|\right]\\
&\quad +(a_4+a_5+a_6)\left[|p_c(\zeta_n,\zeta_{n+1})|+|p_c(\eta_n,\eta_{n+1})|\right]\\
&\quad +(a_7+a_8)\left[|p_c(\zeta_n,\zeta_{n+1})|+|p_c(\eta_n,\eta_{n+1})|\right]\\
&= (a_1+a_2+a_3+a_6+a_8)\left[|p_c(\zeta_{n-1},\zeta_n)|+|p_c(\eta_{n-1},\eta_n)|\right]\\
&\quad +(a_4+a_5+a_6+a_7+a_8)\left[|p_c(\zeta_n,\zeta_{n+1})|+|p_c(\eta_n,\eta_{n+1})|\right]. \qquad (3.4)
\end{aligned}
$$

Set

$$
\begin{aligned}
K_n &= |p_c(\zeta_n,\zeta_{n+1})|+|p_c(\eta_n,\eta_{n+1})|\\
&= A_n+B_n. \qquad (3.5)
\end{aligned}
$$

Then from equations (3.4) and (3.5), we have

$$
\begin{aligned}
K_n &\leq (a_1+a_2+a_3+a_6+a_8)\,K_{n-1}\\
&\quad +(a_4+a_5+a_6+a_7+a_8)\,K_n,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
K_n &\leq \left(\frac{a_1+a_2+a_3+a_6+a_8}{1-a_4-a_5-a_6-a_7-a_8}\right)K_{n-1}\\
&= \sigma\,K_{n-1}, \qquad (3.6)
\end{aligned}
$$

where $\sigma=\left(\frac{a_1+a_2+a_3+a_6+a_8}{1-a_4-a_5-a_6-a_7-a_8}\right)<1$, since $a_1+a_2+a_3+a_4+a_5+2a_6+a_7+2a_8<1$.

Then for each $n\in\mathbb{N}$, we have

$$
K_n\leq\sigma K_{n-1}\leq\sigma^2K_{n-2}\leq\cdots\leq\sigma^nK_0. \qquad (3.7)
$$

If $K_0=0$, then $|p_c(\zeta_0,\zeta_1)|+|p_c(\eta_0,\eta_1)|=0$. Hence, we get $\zeta_0=\zeta_1=F(\zeta_0,\eta_0)$ and $\eta_0=\eta_1=F(\eta_0,\zeta_0)$, which shows that (ζ_0,η_0) is a coupled fixed point of F. Now, we assume that $K_0>0$. For each $n\geq m$, where $n,m\in\mathbb{N}$, by using condition $(\mathcal{CPM}4)$, we have

$$
\begin{aligned}
p_c(\zeta_n,\zeta_m) &\precsim p_c(\zeta_n,\zeta_{n-1})+p_c(\zeta_{n-1},\zeta_{n-2})+\dots\\
&\quad +p_c(\zeta_{m+1},\zeta_m)-p_c(\zeta_{n-1},\zeta_{n-1})-p_c(\zeta_{n-2},\zeta_{n-2})\\
&\quad -\cdots-p_c(\zeta_{m+1},\zeta_{m+1})\\
&\precsim p_c(\zeta_n,\zeta_{n-1})+p_c(\zeta_{n-1},\zeta_{n-2})+\cdots+p_c(\zeta_{m+1},\zeta_m),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|p_c(\zeta_n,\zeta_m)| &\leq |p_c(\zeta_n,\zeta_{n-1})|+|p_c(\zeta_{n-1},\zeta_{n-2})|+\dots\\
&\quad +|p_c(\zeta_{m+1},\zeta_m)|. \qquad (3.8)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
p_c(\eta_n,\eta_m) &\precsim p_c(\eta_n,\eta_{n-1})+p_c(\eta_{n-1},\eta_{n-2})+\dots\\
&\quad +p_c(\eta_{m+1},\eta_m)-p_c(\eta_{n-1},\eta_{n-1})-p_c(\eta_{n-2},\eta_{n-2})\\
&\quad -\cdots-p_c(\eta_{m+1},\eta_{m+1})\\
&\precsim p_c(\eta_n,\eta_{n-1})+p_c(\eta_{n-1},\eta_{n-2})+\cdots+p_c(\eta_{m+1},\eta_m),
\end{aligned}
$$

which implies that

$$\begin{aligned} |p_c(\eta_n, \eta_m)| &\leq |p_c(\eta_n, \eta_{n-1})| + |p_c(\eta_{n-1}, \eta_{n-2})| + \dots \\ &\quad + |p_c(\eta_{m+1}, \eta_m)|. \end{aligned} \tag{3.9}$$

Thus,

$$\begin{aligned} K_n &= |p_c(\zeta_n, \zeta_m)| + |p_c(\eta_n, \eta_m)| \\ &\leq K_{n-1} + K_{n-2} + K_{n-3} + \cdots + K_m \\ &\leq (\sigma^{n-1} + \sigma^{n-2} + \cdots + \sigma^m) K_0 \\ &\leq \Big(\frac{\sigma^m}{1-\sigma}\Big) K_0 \\ &\leq \Big(\frac{\sigma^n}{1-\sigma}\Big) K_0 \to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.10}$$

which implies that $\{\zeta_n\}$ and $\{\eta_n\}$ are Cauchy sequences in $(\mathcal{M}, p_c)$ because $0 \leq \sigma < 1$. Since the complex partial metric space $(\mathcal{M}, p_c)$ is complete, so there exist $u_1, u_2 \in \mathcal{M}$ such that $\zeta_n \to u_1$, $\eta_n \to u_2$ as $n \to \infty$ and

$$p_c(u_1, u_1) = \lim_{n\to\infty} p_c(\zeta_n, u_1) = \lim_{n,m\to\infty} p_c(\zeta_n, \zeta_m) = 0, \tag{3.11}$$

$$p_c(u_2, u_2) = \lim_{n\to\infty} p_c(\eta_n, u_2) = \lim_{n,m\to\infty} p_c(\eta_n, \eta_m) = 0. \tag{3.12}$$

We now show that $u_1 = F(u_1, u_2)$. Suppose on the contrary that $u_1 \neq F(u_1, u_2)$ and $u_2 \neq F(u_2, u_1)$ so that $0 \prec p_c(u_1, F(u_1, u_2)) = H_1$ and $0 \prec p_c(u_2, F(u_2, u_1)) = H_2$, then

$$\begin{aligned} H_1 &= p_c(u_1, F(u_1, u_2)) \\ &\precsim p_c(u_1, \zeta_{n+1}) + p_c(\zeta_{n+1}, F(u_1, u_2)) \\ &\quad - p_c(\zeta_{n+1}, \zeta_{n+1}) \\ &\precsim p_c(u_1, \zeta_{n+1}) + p_c(\zeta_{n+1}, F(u_1, u_2)) \\ &= p_c(\zeta_{n+1}, F(u_1, u_2)) + p_c(u_1, \zeta_{n+1}) \\ &= p_c(F(\zeta_n, \eta_n), F(u_1, u_2)) + p_c(u_1, \zeta_{n+1}) \\ &\precsim a_1\, p_c(\zeta_n, u_1) + a_2\, p_c(\eta_n, u_2) + a_3\, p_c(F(\zeta_n, \eta_n), \zeta_n) \\ &\quad + a_4\, p_c(F(u_1, u_2), u_1) + a_5\, p_c(F(\zeta_n, \eta_n), u_1) \\ &\quad + a_6\, p_c(F(u_1, u_2), \zeta_n) + a_7\, p_c(F(\eta_n, \zeta_n), u_2) \\ &\quad + a_8\, p_c(F(u_2, u_1), \eta_n) + p_c(u_1, \zeta_{n+1}) \\ &= a_1\, p_c(\zeta_n, u_1) + a_2\, p_c(\eta_n, u_2) + a_3\, p_c(\zeta_{n+1}, \zeta_n) \\ &\quad + a_4\, p_c(F(u_1, u_2), u_1) + a_5\, p_c(\zeta_{n+1}, u_1) \\ &\quad + a_6\, p_c(F(u_1, u_2), \zeta_n) + a_7\, p_c(\eta_{n+1}, u_2) \\ &\quad + a_8\, p_c(F(u_2, u_1), \eta_n) + p_c(u_1, \zeta_{n+1}). \end{aligned} \tag{3.13}$$

Passing to the limit as $n \to \infty$ in equation (3.13) and using equations (3.11), (3.12) and condition ($\mathcal{CPM}2$), we obtain

$$\begin{aligned} H_1 &= p_c(u_1, F(u_1,u_2)) \\ &\precsim (a_4+a_6)\, p_c(u_1, F(u_1,u_2)) + a_8\, p_c(u_2, F(u_2,u_1)), \end{aligned}$$

which implies that

$$\begin{aligned} |H_1| &= |p_c(u_1, F(u_1,u_2))| \\ &\leq (a_4+a_6)\, |p_c(u_1, F(u_1,u_2))| + a_8\, |p_c(u_2, F(u_2,u_1))| \\ &= (a_4+a_6)\, |H_1| + a_8\, |H_2|. \end{aligned} \tag{3.14}$$

Similarly, we obtain

$$\begin{aligned} |H_2| &= |p_c(u_2, F(u_2,u_1))| \\ &\leq (a_4+a_6)\, |p_c(u_2, F(u_2,u_1))| + a_8\, |p_c(u_1, F(u_1,u_2))| \\ &= (a_4+a_6)\, |H_2| + a_8\, |H_1|. \end{aligned} \tag{3.15}$$

Hence from equations (3.14) and (3.15), we obtain

$$\begin{aligned} |H_1|+|H_2| &\leq (a_4+a_6)\,(|H_1|+|H_2|) + a_8\,(|H_1|+|H_2|) \\ &= (a_4+a_6+a_8)\,(|H_1|+|H_2|) \\ &\leq (a_1+a_2+a_3+a_4+a_5+2a_6+a_7+2a_8)\,(|H_1|+|H_2|) \\ &< |H_1|+|H_2|, \end{aligned}$$

which is a contradiction, since $a_1+a_2+a_3+a_4+a_5+2a_6+a_7+2a_8 < 1$. Hence, we conclude that $|H_1|+|H_2| = 0$, that is, $|p_c(u_1, F(u_1,u_2))| + |p_c(u_2, F(u_2,u_1))| = 0$ and hence $p_c(u_1, F(u_1,u_2)) = 0$ and $p_c(u_2, F(u_2,u_1)) = 0$. Thus, $F(u_1,u_2) = u_1$ and $F(u_2,u_1) = u_2$. This shows that (u_1,u_2) is a coupled fixed point of F.

Now, we show the uniqueness. Suppose that (v_1,v_2) is another coupled fixed point of F such that $(u_1,u_2) \neq (v_1,v_2)$, then from equation (3.1) and using (3.11), (3.12) and ($\mathcal{CPM}2$), we have

$$\begin{aligned} p_c(u_1,v_1) &= p_c(F(u_1,u_2), F(v_1,v_2)) \\ &\precsim a_1\, p_c(u_1,v_1) + a_2\, p_c(u_2,v_2) + a_3\, p_c(F(u_1,u_2),u_1) \\ &\quad + a_4\, p_c(F(v_1,v_2),v_1) + a_5\, p_c(F(u_1,u_2),v_1) \\ &\quad + a_6\, p_c(F(v_1,v_2),u_1) + a_7\, p_c(F(u_2,u_1),v_2) \\ &\quad + a_8\, p_c(F(v_2,v_1),u_2) \\ &= a_1\, p_c(u_1,v_1) + a_2\, p_c(u_2,v_2) + a_3\, p_c(u_1,u_1) \\ &\quad + a_4\, p_c(v_1,v_1) + a_5\, p_c(u_1,v_1) + a_6\, p_c(v_1,u_1) \\ &\quad + a_7\, p_c(u_2,v_2) + a_8\, p_c(v_2,u_2) \\ &= (a_1+a_5+a_6)\, p_c(u_1,v_1) + (a_2+a_7+a_8)\, p_c(u_2,v_2), \end{aligned}$$

which implies that

$$|p_c(u_1,v_1)| \leq (a_1+a_5+a_6)\, |p_c(u_1,v_1)| + (a_2+a_7+a_8)\, |p_c(u_2,v_2)|. \tag{3.16}$$

By similar fashion, one can obtain

$$|p_c(u_2,v_2)| \leq (a_1+a_5+a_6)\, |p_c(u_2,v_2)| + (a_2+a_7+a_8)\, |p_c(u_1,v_1)|. \tag{3.17}$$

Hence from equations (3.16) and (3.17), we obtain

$$\begin{aligned}
|p_c(u_1,v_1)|+|p_c(u_2,v_2)| &\leq (a_1+a_5+a_6)\,[|p_c(u_1,v_1)|+|p_c(u_2,v_2)|] \\
&\quad +(a_2+a_7+a_8)\,[|p_c(u_1,v_1)|+|p_c(u_2,v_2)|] \\
&= (a_1+a_2+a_5+a_6+a_7+a_8)\times \\
&\quad [|p_c(u_1,v_1)|+|p_c(u_2,v_2)|] \\
&\leq (a_1+a_2+a_3+a_4+a_5+2a_6+a_7+2a_8)\times \\
&\quad [|p_c(u_1,v_1)|+|p_c(u_2,v_2)|] \\
&< |p_c(u_1,v_1)|+|p_c(u_2,v_2)|,
\end{aligned}$$

which is a contradiction, since $a_1+a_2+a_3+a_4+a_5+2a_6+a_7+2a_8 < 1$. Hence, we conclude that $|p_c(u_1,v_1)|+|p_c(u_2,v_2)| = 0$, that is, $p_c(u_1,v_1) = 0$ and $p_c(u_2,v_2) = 0$ and hence $u_1 = v_1$ and $u_2 = v_2$. This shows that the coupled fixed point of F is unique. This completes the proof. $\square$

4. Consequences of Theorem 3.1

By taking $a_1 = k$, $a_2 = l$ and $a_3 = a_4 = \cdots = a_8 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.1. *Let $(\mathcal{M}, p_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ satisfying the following contractive condition for all $\zeta,\eta,\theta,\lambda\in\mathcal{M}$:*

$$p_c(F(\zeta,\eta),F(\theta,\lambda)) \leq k\,p_c(\zeta,\theta)+l\,p_c(\eta,\lambda), \tag{4.1}$$

where k, l are nonnegative constants such that $k+l<1$. Then F has a unique coupled fixed point.

By taking $a_3 = k$, $a_4 = l$ and $a_1 = a_2 = a_5 = \cdots = a_8 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.2. *Let $(\mathcal{M}, p_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ satisfying the following contractive condition for all $\zeta,\eta,\theta,\lambda\in\mathcal{M}$:*

$$p_c(F(\zeta,\eta),F(\theta,\lambda)) \leq k\,p_c(F(\zeta,\eta),\zeta)+l\,p_c(F(\theta,\lambda),\theta), \tag{4.2}$$

where k, l are nonnegative constants such that $k+l<1$. Then F has a unique coupled fixed point.

By taking $a_5 = k$, $a_6 = l$ and $a_1 = a_2 = a_3 = a_4 = a_7 = a_8 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.3. *Let $(\mathcal{M}, p_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ satisfying the following contractive condition for all $\zeta,\eta,\theta,\lambda\in\mathcal{M}$:*

$$p_c(F(\zeta,\eta),F(\theta,\lambda)) \leq k\,p_c(F(\zeta,\eta),\theta)+l\,p_c(F(\theta,\lambda),\zeta), \tag{4.3}$$

where k, l are nonnegative constant with $k+2l<1$. Then F has a unique coupled fixed point.

Remark 4.4. (1) Theorem 3.1 extends the results of Aydi [1] from partial metric space to the setting of complex partial metric space.

(2) Theorem 3.1 also extends the results of Sabetghadam *et al.* [17] from cone metric space to the setting of complex partial metric space.

(3) Corollary 4.1, Corollary 4.2 and Corollary 4.3 extend Theorem 2.1, Theorem 2.4 and Theorem 2.5 respectively of Aydi [1] from partial metric space to the setting of complex partial metric space.

Now, we give an example to validate the result.

Example 4.5. Let $\mathcal{M} = [0, +\infty)$ endowed with the usual complex partial metric p_c defined by $p_c\colon \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ with $p_c(\zeta, \eta) = \max\{\zeta, \eta\}(1 + i)$. The complex partial metric space $(\mathcal{M}, p_c)$ is complete because $(\mathcal{M}, p_c^t)$ is complete. Indeed, for any $\zeta, \eta \in \mathcal{M}$,

$$\begin{aligned} p_c^t(\zeta, \eta) &= 2p_c(\zeta, \eta) - p_c(\zeta, \zeta) - p_c(\eta, \eta) \\ &= 2\max\{\zeta, \eta\}(1 + i) - (\zeta + i\zeta) - (\eta + i\eta) \\ &= |\zeta - \eta| + i|\zeta - \eta| = |\zeta - \eta|(1 + i). \end{aligned} \tag{4.4}$$

Thus, $(\mathcal{M}, p_c^t)$ is the Euclidean complex metric space which is complete. Consider the mapping $F\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ defined by $F(\zeta, \eta) = \frac{\zeta+\eta}{12}$. Now, for any $\zeta, \eta, \theta, \lambda \in \mathcal{M}$, we have

(1)

$$\begin{aligned} p_c(F(\zeta, \eta), F(\theta, \lambda)) &= \frac{1}{12}\max\{\zeta + \eta, \theta + \lambda\}(1 + i) \\ &\leq \frac{1}{12}[\max\{\zeta, \theta\}(1 + i) + \max\{\eta, \lambda\}(1 + i)] \\ &= \frac{1}{12}[p_c(\zeta, \theta) + p_c(\eta, \lambda)], \end{aligned}$$

which is the contractive condition of Corollary 4.1 for $k = 1/6 < 1$ (if $k = l$). Therefore, by Corollary 4.1, F has a unique coupled fixed point, which is $(0, 0)$.

(2)

$$\begin{aligned} p_c(F(\zeta, \eta), F(\theta, \lambda)) &= \frac{1}{12}\max\{\zeta + \eta, \theta + \lambda\}(1 + i) \\ &\leq \frac{1}{12}[\max\{\zeta + \eta, \zeta\}(1 + i) + \max\{\theta + \lambda, \theta\}(1 + i)] \\ &= \frac{1}{12}[p_c(F(\zeta, \eta), \zeta) + p_c(F(\theta, \lambda), \theta)], \end{aligned}$$

which is the contractive condition of Corollary 4.2 for $k = 1/6 < 1$ (if $k = l$). Therefore, by Corollary 4.2, F has a unique coupled fixed point, which is $(0, 0)$.

(3)

$$
\begin{aligned}
p_c(F(\zeta,\eta),F(\theta,\lambda)) &= \frac{1}{12}\max\{\zeta+\eta,\theta+\lambda\}(1+i)\\
&\le \frac{1}{12}[\max\{\zeta+\eta,\theta\}(1+i)+\max\{\theta+\lambda,\zeta\}(1+i)]\\
&= \frac{1}{12}[p_c(F(\zeta,\eta),\theta)+p_c(F(\theta,\lambda),\zeta)],
\end{aligned}
$$

which is the contractive condition of Corollary 4.3 for $k=1/6<2/3<1$ (if $k=l$). Therefore, by Corollary 4.3, F has a unique coupled fixed point, which is $(0,0)$.

Note that if the mapping $F\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is given by $F(\zeta,\eta)=\frac{\zeta+\eta}{2}$, then F satisfies contractive condition of Corollary 4.1, 4.2, 4.3 for $k=1$ (if $k=l$),

$$
\begin{aligned}
p_c(F(\zeta,\eta),F(\theta,\lambda)) &= \frac{1}{2}\max\{\zeta+\eta,\theta+\lambda\}(1+i)\\
&\le \frac{1}{2}[\max\{\zeta,\theta\}(1+i)+\max\{\eta,\lambda\}(1+i)]\\
&= \frac{1}{2}[p_c(\zeta,\theta)+p_c(\eta,\lambda)].
\end{aligned}
$$

In this case $(0,0)$ and $(1,1)$ are both coupled fixed points of F, and hence, the coupled fixed point of F is not unique. This shows that the condition $k<1$ in Corollary 4.1, 4.2 and hence $k+l<1$ and the condition $k<2/3$ in Corollary 4.3 and hence $k+2l<1$ cannot be omitted in the statement of the aforesaid results.

5. Conclusion

In this paper, we establish some coupled fixed point theorems for contractive type conditions in the setting of complex partial metric spaces and give some consequences of the main result. The results of findings in this paper extend, generalize and enrich several results from the current existing literature.

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H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India.
Email address: saluja1963@gmail.com