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ON POWER-ASSOCIATIVE NILALGEBRAS OF DIMENSION N AND NILINDEX $\geq N-3$

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ABSTRACT. The present study is about commutative power-associative nilalgebras of dimension $n \geq 5$ and nilindex $n-3$, over a field K. We prove the solvability of such algebras when K is of characteristic 0 or sufficiently large.

1. INTRODUCTION

It is well known that every finite-dimensional Jordan nilalgebra over a field of characteristic $\neq 2$ is nilpotent [14]. In [1], Albert asked if the same occurs in the variety of commutative power-associative algebras. This question was answered by Suttles [13], who constructed a 5-dimensional algebra which is commutative power-associative, nil of index 4, but it is not nilpotent. However, this algebra is solvable. Since then, a modified version of Albert's question was formulated as following:

Albert's Problem. Is every finite-dimensional commutative power-associative nilalgebra solvable?

This question remains open but in some particular cases, a positive answer was obtained relative to the nilindex and dimension [8, 9, 2]. The following results are prove in [5]: commutative power-associative nilalgebra of dimension n and nilindex $n-1$ or $n-2$ are solvable; commutative power-associative nilalgebras of dimension 7 are solvable. In [7], it was proved that any commutative power-associative nilalgebras of dimension n and nilindex $n-3$, over the field of complex numbers $\mathbb{C}$, is solvable. In this paper, we generalize this result to any field K with characteristic 0 or sufficiently large.

2. PRELIMINARIES

Let A be a commutative algebra over a field K of characteristic $\neq 2, 3$. We have different ways to define the powers of an element x in A. The principal powers of an element x in A are defined inductively by:

$$x^1 = x;\ x^{k+1} = x^k x,\ \forall k \geq 1.$$

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We also define the plenary powers of an element x of the algebra A by:

$$x^{[1]} = x;\ x^{[k+1]} = x^{[k]}x^{[k]},\ \forall k \geq 1.$$

The principal powers of A are defined inductively by:

$$A^1 = A,$$

$$A^k = \sum_{i=1}^{k-1} A^i A^{k-i},\ \forall k \geq 2.$$

We also define the plenary powers of A by:

$$A^{[1]} = A,\ A^{[k+1]} = A^{[k]}A^{[k]}, \forall k \geq 1.$$

A commutative algebra A is called a *power-associative* algebra if for any $a \in A$, the subalgebra $K[a]$ of A generated by a is associative. This is equivalent to defining, for any $a \in A$ and all integers $i, j \geq 1$,

$$a^i a^j = a^{i+j}.$$

An element $x \in A$ is called nilpotent of index $\leq n$ if $x^n = 0$. We say that the algebra A is a nilalgebra if every element x of A is nilpotent. The smallest positive integer n such that every element of the algebra is nilpotent of index $\leq n$ is called the nilpotent nilindex of the algebra.

A nilpotent algebra is an algebra for which there is a natural number n such that any product containing at least n elements of the algebra is zero. Thus, an algebra A is nilpotent of index n, if $A^n = 0$ and $A^{n-1} \neq 0$. The algebra A is solvable of index n if $A^{[n]} = 0$ and $A^{[n-1]} \neq 0$. We note that any nilpotent algebra is solvable.

In [11, 12], we have:

Definition 2.1. An algebra A over a field K is called strictly power-associative in case every scalar extension $A_F = F \otimes_K A$ is power-associative, for any extension F of K.

Proposition 2.2. *Let A be a power-associative algebra over a field K of characteristic $\neq 2, 3$ and 5. Then A is strictly power-associative.*

In [4], we have:

Definition 2.3. Two commutative rings R_1 and R_2 are called elementarely equivalent in case for any assumption σ,

σ is valid over R_1 if and only if σ is valid over R_2.

Proposition 2.4. *Two algebraicaly closed fields are elementarely equivalent if and only if they have a same characteristic.*

Lemma 2.5. *If an assumption σ is true over every fields of characteristic 0, then there exists a prime number q such that σ is true over every fields of characteristic greater or equal to q.*

In [14], we have:

Theorem 2.6. *Every finite-dimensional Jordan commutative nilalgebra, over a field K of characteristic $\neq 2$, is nilpotent.*

In [7], we get:

Theorem 2.7. *Let A be a commutative power-associative nilalgebra with dimension n and nilindex $\geq n-3$, over the field of complex numbers. Then A is solvable.*

From [6], we get following results:

Theorem 2.8. *Let A be a commutative n-dimensional power-associative nilalgebra with nilindex $\geq n-2$, over a field K of characteristic 0 or sufficiently large. Then A is solvable.*

Theorem 2.9. *Let A be a commutative power-associative nilalgebra of nilindex $n-1$ and dimension $n \geq 4$, over a field K of characteristic 0 or sufficiently large. Then either A is isomorphic to the Suttle's algebra or A is nilpotent of index $n-1$.*

Theorem 2.10. *Let A be a commutative power-associative nilalgebra of nilindex n and dimension n, over a field K of characteristic 0 or sufficiently large, then A is nilpotent of index n.*

We also have:

Proposition 2.11. *Every commutative power-associative nilalgebra of dimension n and nilindex $n+1$ is nilpotent of index $n+1$.*

Proof. Let A be commutative power-associative nilalgebra of dimension n and nilindex $n+1$, over a field K and x an element of A with maximal nilindex. We have $x^n \neq 0$ and $x^{n+1} = 0$. The family $\{x, x^2, ..., x^n\}$, containing n elements of A, is then linearly independent. Thus, it is a basis of A. We get $A = K[x]$. This implies that the algebra A is nilpotent of index $n+1$. □

Proposition 2.12. *Let A be a n-dimensional commutative power-associative nilalgebra with nilindex k. Then $k \leq n+1$.*

In the following, we shall denote by $\langle a_1, a_2, ..., a_s\rangle$ the subspace of the algebra A generated by the elements $a_1, a_2, ..., a_s$ of A.

3. Generalization of the binomial formula

Let A be a commutative and not necessarily associative algebra, over a field K. In this section, we establish a multinomial formula which is a generalization of the binomial formula obtained by Guzzo [10] in the case of non-associative and commutative algebras.

Let $m \geq 1$ be an integer. Suppose $f_i : A \longrightarrow A$ arbitrary functions, with $i \in \{1, ..., m\}$. We define $(f_1, ..., f_m) : A \oplus ... \oplus A \longrightarrow A$ by:

(a) $(f_1) = f_1$,

(b) $$(f_1, ..., f_m)(a_1, ..., a_m) = \frac{1}{m}\sum_{i=0}^{m-1}(f_1, ..., f_{m-1})(a_{\sigma^i(1)}, ..., a_{\sigma^i(m-1)})f_m(a_{\sigma^i(m)}),$$

where $(a_1, ..., a_m) \in A \oplus ... \oplus A$ and σ is the cycle $(1\ 2\ ...\ m)$ in the symmetric group S_m. In [10], we have following results:

Lemma 3.1. *If for any $i \in \{1, ..., m\}$, f_i is a linear function, then $(f_1, ..., f_m)$ is a m-linear symmetric function of m variables in $A \oplus ... \oplus A$ with values in A.*

Let $f; g : A \longrightarrow A$ as above, $a, b \in A$ and $m, p \in \mathbb{N}^*$. We use the following notations:

(a) $f^{(1)} = f$; $f^{(m)} = (f, ..., f)$; $(f^{(m)}; g^{(p)}) = (f, ..., f, g, ..., g)$;
(b) $(f^{(0)}, g^{(p)}) = g^{(p)}$; $(f^{(m)}, g^{(0)}) = f^{(m)}$;
(c) $a^{(1)} = a$; $a^{(m)} = (a, ..., a)$; $(a^{(m)}, b^{(p)}) = (a, ..., a, b, ..., b)$;
(d) $(a^{(m)}, b^{(0)}) = a^{(m)}$; $(a^{(0)}, b^{(p)}) = b^{(p)}$.

Now, let us give the binomial formula obtained by Guzzo [10]:

Lemma 3.2. *Let A be an algebra over K, $a, b \in A$ and $m \in \mathbb{N}^*$. We have:*

$$(a+b)^m = \sum_{i=0}^{m} \binom{m}{i} id_A^{(m)}(a^{(m-i)}, b^{(i)}),$$

where id_A is the identity function on A.

Let i, $i_1, \cdots, i_k$ be integers such that $i_1 + \cdots + i_k = i$. Let us note :

$$\binom{i}{i_1, ..., i_k} := \frac{i!}{i_1! \cdots i_k!}.$$

We get:

Lemma 3.3. *Let $\{a_1, ..., a_k, b\}$ be a family with $k+1$ elements of A and $n \geq 2$ be an integer. For any integers $i, j \geq 1$ such that $n = i + j$, we have:*

$$id_A^{(n)}((a_1 + ... + a_k)^{(i)}, b^{(j)}) = \sum_{i_1+...+i_k=i} \binom{i}{i_1, ..., i_k} id^{(n)}(a_1^{(i_1)}, ..., a_k^{(i_k)}, b^{(j)}).$$

Proof. By induction on i, we have:
For $i = 1$, we get $j = n - 1$ and:

$$\begin{aligned} id_A^{(n)}((a_1 + ... + a_k)^{(1)}, b^{(j)}) &= id_A^{(n)}(a_1 + ... + a_k, b^{(j)}) \\ &= id_A^{(n)}(a_1^{(1)}, b^{(j)}) + id_A^{(n)}(a_2^{(1)}, b^{(j)}) + ... + id_A^{(n)}(a_k^{(1)}, b^{(j)}) \\ &= \sum_{i_1+...+i_k=1} \binom{1}{i_1, ..., i_k} id_A^{(n)}(a_1^{(i_1)}, ..., a_k^{(i_k)}, b^{(j)}) \end{aligned}$$

The result is true for $i = 1$. Let us suppose it is true up to the order i and show that it is true to the order $i + 1$. To the order $i + 1$, we get $j = n - i - 1$

and:

$$\begin{aligned}
id_A^{(n)}((a_1+...+a_k)^{(i+1)}, b^{(j)}) &= id_A^{(n)}((a_1+...+a_k)^{(i)}, (a_1+...+a_k), b^{(j)}) \\
&= \sum_{p=1}^{k} id_A^{(n)}((a_1+...+a_k)^{(i)}, a_p, b^{(j)}) \\
&= \sum_{p=1}^{k} \left(\sum_{i_1+...+i_k=i} \binom{i}{i_1,...,i_k} id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}, a_p^{(1)}, b^{(j)}) \right) \\
&= \sum_{i_1+...+i_k=i} \left(\sum_{p=1}^{k} \binom{i}{i_1,...,i_k} id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}, a_p^{(1)}, b^{(j)}) \right) \\
&= \sum_{i_1+...+i_k=i+1} \left(\sum_{p=1}^{k} \frac{i!}{(i_p-1)\prod_{\substack{1\le j\le k\\ j\ne p}} i_j!} \right) id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}, b^{(j)}).
\end{aligned}$$

We have:

$$\binom{i+1}{i_1,...,i_k} = \frac{i!}{(i_1-1)!i_2!...i_k!} + ... + \frac{i!}{i_1!...i_{k-1}!(i_k-1)!}.$$

Then,

$$id_A^{(n)}((a_1+...+a_k)^{(i+1)}, b^{(j)}) = \sum_{i_1+...+i_k=i+1} \binom{i+1}{i_1,...,i_k} id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}, b^{(j)}).$$

The result being true to the order $i+1$, we deduce that it is true for every integer $i \geq 1$. □

We deduce the following multinomial formula:

Lemma 3.4. *Let A be a nonassociative and commutative algebra, over a field K, $(a_1,...,a_k) \in A^k$ and $n \geq 1$. Then:*

$$(a_1+...+a_k)^n = \sum_{i_1+...+i_k=n} \binom{n}{i_1,...,i_k} id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}),$$

where id_A is the identity function on A.

Proof. From the binomial formula, we get:

$$\begin{aligned}
(a_1+...+a_k)^n &= ((a_1+...+a_{k-1})+a_k)^n \\
&= \sum_{i+i_k=n} \binom{n}{i, i_k} id_A^{(n)}((a_1+...+a_{k-1})^{(i)}, a_k^{(i_k)})
\end{aligned}$$

By Lemma 3.3, we have:

$$id_A^{(n)}((a_1+...+a_{k-1})^{(i)}, a_k^{(i_k)}) = \sum_{i_1+...+i_{k-1}=i} \binom{i}{i_1,...,i_{k-1}} id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}).$$

Then,

$$
\begin{aligned}
(a_1+...+a_k)^n &= \sum_{i+i_k=n}\binom{n}{i,i_k}\Bigg(\sum_{i_1+...+i_{k-1}=i}\binom{i}{i_1,...,i_{k-1}}id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)})\Bigg)\\
&= \sum_{i+i_k=n}\Bigg(\sum_{i_1+...+i_{k-1}=i}\binom{n}{i,i_k}\times\binom{i}{i_1,...,i_{k-1}}id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)})\Bigg)\\
&= \sum_{i+i_k=n}\Bigg(\sum_{i_1+...+i_{k-1}=i}\binom{n}{i_1,...,i_k}id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)})\Bigg)\\
(a_1+...+a_k)^n &= \sum_{i_1+...+i_k=n}\binom{n}{i_1,...,i_k}id_A^{(n)}(a_1^{(i_1)},...,a_k^{(i_k)}).
\end{aligned}
$$

□

4. Recalls on scalar extensions

Let A and B be algebras over a field K. We consider the tensor product $A\otimes_K B$ of the vector spaces A and B, where all elements of $A\otimes_K B$ are finite sums $\Sigma a_i\otimes_K b_i$ with $a_i\in A$ and $b_i\in B$.

Multiplication is defined by:

$$(a\otimes_K b)(a^{'}\otimes_K b^{'}) = aa^{'}\otimes_K bb^{'} \text{ with } a,a^{'}\in A \text{ and } b,b^{'}\in B.$$

Consequently, we have:

$$\Bigg(\sum_i a_i\otimes_K b_i\Bigg)\Bigg(\sum_j a_j^{'}\otimes_K b_j^{'}\Bigg) = \sum_{i,j} a_ia_j^{'}\otimes_K b_ib_j^{'}$$

with $a_i, a_j^{'}\in A$ and $b_i, b_j^{'}\in B$.

We also have:

$$\alpha\Bigg(\sum_i a_i\otimes_K b_i\Bigg) = \sum_i(\alpha a_i)\otimes_K b_i = \sum_i a_i\otimes_K(\alpha b_i)$$

with $\alpha\in K$, $a_i\in A$ and $b_i\in B$.

Thus, the tensor product $A\otimes_K B$ is an algebra over K. Particularily, in the case where F is an arbitrary extension of K, we set $A_F = F\otimes_K A$. Note that A_F is an algebra over the field F and is called the *scalar extension* of A.

In the following, let A be a commutative power-associative nilalgebra of dimension $n\geq 5$ and nilindice $n-3$ over a field K of characteristic 0 and $\overline{K}$ the algebraic closure of K. The field $\overline{K}$ is then an algebraicaly closed field of characteristic 0 and also is an algebra over K. Now, let us consider the scalar extension $A_{\overline{K}} = \overline{K}\otimes_K A$.

Proposition 4.1. *$A_{\overline{K}}$ is a commutative algebra over the field $\overline{K}$.*

Proof. We already know that $A_{\overline{K}}$ is an algebra over $\overline{K}$. Let x,y be elements in $A_{\overline{K}}$. Then $x = \sum_{i=1}^{n}\alpha_i\otimes_K a_i$ and $y = \sum_{j=1}^{m}\beta_j\otimes_K b_j$, with $\alpha_i,\beta_j\in\overline{K}$ and

$a_i, b_j \in A$. Using commutativity of A and some properties of the tensor product, we demonstrate that $xy = yx$. This prove the commutativity of $A_{\overline{K}}$ over $\overline{K}$. □

We have the following result:

Proposition 4.2. *$A_{\overline{K}}$ is a power-associative algebra over $\overline{K}$.*

Proof. We know that $A_{\overline{K}}$ is an algebra over $\overline{K}$ which is a field of characteristic 0. Proposition 2.2 implies that the algebra A is a strictly power-associative algebra. Then $A_{\overline{K}}$ is a power-associative algebra over $\overline{K}$. □

In [3], we have the following result.

Proposition 4.3. *Let K be a field and L an extension of K. If E is a vector space over K, then $dim_L(E_L) = dim_K(E)$ where $E_L = L \otimes_K E$ and if $(e_i)_{i\in I}$ is a basis of E over K, then $(1 \otimes_K e_i)_{i\in I}$ is a basis of E_L over L.*

Thus, $dim_{\overline{K}} A_{\overline{K}} = dim_K A = n$ and if $\mathcal{B} = \{e_1, ..., e_n\}$ is a basis of A over K, then the family $\{1 \otimes_K e_1, ..., 1 \otimes_K e_n\}$ is also a basis of $A_{\overline{K}}$ over $\overline{K}$.

Proposition 4.4. *The mapping*

$$\varphi : A \longrightarrow K \otimes_K A; a \longmapsto 1 \otimes_K a,$$

is an isomorphism of K-algebras.

Proof. Firstly, we prove that φ is a well defined surjection morphism of K-algebras and secondly we conclude using Proposition 4.3.

□

Now, we are going to prove the following result:

Proposition 4.5. *If K is a field containing at least $n-2$ distinct elements, then $A_{\overline{K}}$ is nil of nilindex $n-3$.*

Proof. The K-algebra A is nil of nilindex $n-3$ and dimension n. We want to prove that $A_{\overline{K}}$ is nil of nilindex $n-3$ as a $\overline{K}$-algebra.

Let $(\lambda_1, ..., \lambda_n) \in K^n$, such that $\lambda_1 \neq 0$. We have:

$$(\lambda_1 e_1 + ... + \lambda_n e_n)^{n-3} = 0.$$

By Lemma 3.4, we know that:

$$(\lambda_1 e_1 + ... + \lambda_n e_n)^{n-3} = \sum_{i_1+...+i_n=n-3} \binom{n-3}{i_1, ..., i_n} id_A^{(n-3)}((\lambda_1 e_1)^{(i_1)}, ..., (\lambda_n e_n)^{(i_n)}).$$

We get:

$$(\lambda_1 e_1 + ... + \lambda_n e_n)^{n-3} = \sum_{i_1+...+i_n=n-3} \binom{n-3}{i_1, ..., i_n} \lambda_1^{i_1} ... \lambda_n^{i_n} id_A^{(n-3)}(e_1^{(i_1)}, ..., e_n^{(i_n)}).$$

Thus:

$$\sum_{i_1+...+i_n=n-3} \binom{n-3}{i_1, ..., i_n} \lambda_1^{i_1} ... \lambda_n^{i_n} id_A^{(n-3)}(e_1^{(i_1)}, ..., e_n^{(i_n)}) = 0.$$

Let $k \in \{0, ..., n-3\}$ be an integer, and set:

$$f_k(e_1, ..., e_n) = \sum_{i_2+...+i_n=n-3-k} \binom{n-3}{k, i_2, ..., i_n} \lambda_2^{i_2} ... \lambda_n^{i_n} id_A^{(n)}(e_1^{(k)}, e_2^{(i_2)}, ..., e_n^{(i_n)}).$$

Identity (1) implies the following relation:

$$\sum_{k=0}^{n-3} \lambda_1^k f_k(e_1, ..., e_n) = 0.$$

As K contains at least $n-2$ distinct elements, there is a family $\{\beta_1, ..., \beta_{n-2}\}$ of elements of K, such that for all $i \neq j$ we get $\beta_i \neq \beta_j$. When λ_1 covers $\{\beta_1, ..., \beta_{n-2}\}$, we obtain:

$$BX = 0, \tag{$\star$}$$

where

$$B = \begin{pmatrix} 1 & \beta_1 & \cdots & \beta_1^{n-3} \\ 1 & \beta_2 & \cdots & \beta_2^{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \beta_{n-2} & \cdots & \beta_{n-2}^{n-3} \end{pmatrix} \text{ and } X = \begin{pmatrix} f_0(e_1, ..., e_n) \\ f_1(e_1, ..., e_n) \\ \vdots \\ f_{n-3}(e_1, ..., e_n) \end{pmatrix}.$$

The determinant of the matrice B is a Vandermonde's one:

$$det(B) = \prod_{1 \leq i < j \leq n-2} (\beta_j - \beta_i).$$

We have $det(B) \neq 0$ because $\beta_i \neq \beta_j$ for all $i \neq j$. Then, the matrice B is inversible. From ($\star$), we get $X = 0$. We deduce that for all $k \in \{0, ..., n-3\}$, we have

$$f_k(e_1, ..., e_n) = 0.$$

Otherwise, for all $z \in A_{\overline{K}} = \overline{K} \otimes_K A$, we have: $z = \displaystyle\sum_{i=1}^{n} \alpha_i 1 \otimes_K e_i$ with $\alpha_i \in \overline{K}$, then:

$$\begin{aligned} z^{n-3} &= (\alpha_1 1 \otimes_K e_1 + ... + \alpha_n 1 \otimes_K e_n)^{n-3} \\ &= \sum_{i_1+...+i_n=n-3} \binom{n-3}{i_1, ..., i_n} \alpha_1^{i_1} ... \alpha_n^{i_n} 1 \otimes_K id_A^{(n-3)}(e_1^{(i_1)}, ..., e_n^{(i_n)}) \\ &= 1 \otimes_K \left(\sum_{i_1+...+i_n=n-3} \binom{n-3}{i_1, ..., i_n} \alpha_1^{i_1} ... \alpha_n^{i_n} id_A^{(n-3)}(e_1^{(i_1)}, ..., e_n^{(i_n)}) \right) \\ &= 1 \otimes_K \left(\sum_{k=0}^{n-3} \alpha_1^k f_k(e_1, ..., e_n) \right) \\ &= 0. \end{aligned}$$

Thus, for all $z \in A_{\overline{K}}$, we have $z^{n-3} = 0$.

By Proposition 4.4, we know that $K \otimes_K A$ is a nilalgebra of nilindex $n-3$ since the K-algebras A and $K \otimes_K A$ are isomorphic. Then, there exists $z_0 \in K \otimes_K A$ such that $z_0^{n-3} = 0$ and $z_0^{n-4} \neq 0$. Since $K \otimes_K A \subseteq \overline{K} \otimes_K A$, then $z_0 \in A_{\overline{K}}$. Thus,

$n-3$ is the smallest natural number such that for all $x \in A_{\overline{K}}$, $x^{n-3} = 0$. Then $A_{\overline{K}}$ is nil of nilindex $n-3$ as a $\overline{K}$-algebra.

$\square$

5. Main result

Let A be a commutative power-associative nilalgebra of dimension $n \geq 5$ and nilindex $n-3$, over a field K of characteristic 0. We have:

(1) If $n = 5$, then the nilindex of A is equal to 2. Then A is a Jordan algebra and by Theorem 2.6, we conclude that it is nilpotent.
(2) If $n = 6$, then the nilindex of A is equal to 3. For all $x \in A$, we have $x^3 = 0$. Linearizing this identity, we get $2(yx)x + yx^2 = 0$, for all $x, y \in A$. Replacing y by yx, we get $2((yx)x)x + (yx)x^2 = 0$. Otherwise, multiplying the same egality by x, we get $2((yx)x)x + (yx^2)x = 0$. Thus $(yx^2)x = (yx)x^2$. This prove that A is a Jordan algebra. By Theorem 2.6, we conclude that A is nilpotent.
(3) If $n = 7$, then the nilindex of A is equal to 4. Using Suttle's algebra [13], we construct a 7-dimensional commutative power-associative nilalgebra of nilindex 4 which is not nilpotent. However it is solvable.

Example 5.1. Let K be a field of characteristic 0 and A be a commutative algebra of dimmension 7 with basis $\mathcal{B} = \{e_1, ..., e_7\}$ and nonzero products given by $e_1e_2 = e_2e_4 = -e_1e_5 = e_3$, $e_1e_3 = e_4$, $e_2e_3 = e_5$. Then $e_6A = e_7A = 0$ and we get

$$A = \langle e_1, ..., e_7\rangle, A^2 = \langle e_3, e_4, e_5\rangle, A^3 = \langle e_3, e_4, e_5\rangle = A^2.$$

We know that $A^{[3]} = A^2A^2 = 0$, and A is a power-associative nilalgebra of nilindex 4 which is not nilpotent but solvable.

Thus, for $n \in \{5, 6\}$, A is nilpotent and for $n \geq 7$, it is not necessarily nipotent but solvable.

In the section 4, we have proved that if $\overline{K}$ is the algebraic closure of K, then the scalar extension $A_{\overline{K}}$ is also a commutative power-associative nilalgebra of dimension n and nilindex $n-3$ over the field $\overline{K}$. By Proposition 2.4, we know that Theorem 2.7 is valid over every algebraicaly closed fields of chacteristic 0. Then $A_{\overline{K}}$ is solvable as a $\overline{K}$-algebra. We have the following result:

Theorem 5.2. *Let A be a commutative power-associative nilalgebra of dimension $n \geq 5$ and nilindex $n-3$, over a field K of characteristic 0 or sufficiently large. Then, A is nilpotent for $n = 5$ or 6, and solvable for $n \geq 7$.*

Proof. Firstly, let us suppose that K is a field of characteristic 0. We have already proved that if $n \in \{5, 6\}$, then A is nilpotent.

Now, suppose that $n \geq 7$. By next results, we know that if A is a commutative power-associative nilalgebra of dimension n and nilindex $n-3$, over a field K of characteristic 0, then $A_{\overline{K}}$ is solvable as a $\overline{K}$-algebra. Since $K \subseteq \overline{K}$ (as a subfield), then $A_{\overline{K}}$ is solvable as a K-algebra. Thus, there exists an natural integer $s \geq 2$ such that $A_{\overline{K}}^{[s]} = 0$.

Considering the isomorphism φ as defined in Propositon 4.4, we have

$$\varphi(A) = K \otimes_K A \subseteq \overline{K} \otimes_K A.$$

Since $A_{\overline{K}}^{[s]} = 0$, then

$$\begin{aligned} A^{[s]} &= (\varphi^{-1}(K \otimes_K A))^{[s]} \\ &= \varphi^{-1}((K \otimes_K A)^{[s]}) \\ &= 0, \end{aligned}$$

because $(K \otimes_K A)^{[s]} \subseteq A_{\overline{K}}^{[s]} = 0$. Thus, A is solvable if K is of characteristic 0.

In the case where K is of characteristic $\neq 0$, by Lemma 2.5, we deduce that there exists a prime number q such that the Theorem remains valid when the characteristic of K is $\geq q$, i.e sufficiently large.

By Theorems 2.8, 2.9, 2.10 and Proposition 2.11, we deduce that the result is also true in the case where the nilindex of A is $\geq n-3$. □

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