

SOME FIXED POINT THEOREMS ON ALMOST GENERALIZED $\alpha - \beta - \psi - \varphi - \vartheta$ CONTRACTIVE MAPPINGS IN PARTIAL METRIC SPACES

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ABSTRACT. In this paper, we introduced the notion of almost generalized $\alpha - \beta - \psi - \varphi - \vartheta$ contractive mappings and proved the existence and uniqueness of fixed points for such mappings in the setting of partial metric spaces. Our results extend and strengthen various known results in this direction. Furthermore, an example is presented to illustrate our result.

1. INTRODUCTION AND PRELIMINARIES

Matthews [2] presented a very interesting generalization of the metric space known as partial metric space (PMS) in which the condition $d(x, x) = 0$ is replaced by the condition $d(x, x) \leq d(x, y)$ and proved the partial metric version of Banach fixed point theorem.

Alber and Delabriere [3] introduced the concept of weak contraction and the generalization of φ -weak contraction was defined by Zhang and Song [4]. Afterward, Dutta and Choudhury [5] introduced $(\psi - \varphi)$ weak contraction as an extension of weak contraction in complete metric spaces. Further, Samet et al. [6] extended and generalized the Banach contraction principle by introducing a new class of contractive type mappings known as $\alpha - \psi$ contractive type mappings. Karapinar and Samet [7] generalized the $\alpha - \psi$ contractive type mappings and established various fixed point theorems. Also Karapinar et al. [14] shown some results in partial metric spaces using rational type expressions. Recently, Karapinar et al. [8] generalized the concept of $\alpha - (\psi - \varphi)$ Contractive mappings on quasi-partial metric spaces. Furthermore, Aydi H. et al. [9] considered almost generalized contraction in setting of partial metric spaces and established some fixed point theorems.

Definition 1.1. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied :

(a) ψ is non decreasing and continuous;

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(b) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.2. We denote Ψ set of altering distance functions.

Definition 1.3. Let Θ be the family of functions $\vartheta : [0, \infty) \rightarrow [0, \infty)$ such that

- (i) ϑ is continuous;
- (ii) $\vartheta(t) = 0$ if and only if $t = 0$.

Dutta and Choudhury [5] introduced $(\psi - \varphi)$ weak contraction and established fixed point theorem as follows:

Theorem 1.4. [5] *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a self-map satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \quad (1.1)$$

for all $x, y \in X$, where $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ are both continuous and nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X .

Recently, Karapinar et al.[14] proved some fixed point theorems for mappings satisfying rational type contractive condition using auxiliary functions in partial metric spaces.

Theorem 1.5. [14] *Let (X, ρ) be a complete partial metric space and $T : X \rightarrow X$ be a self-map satisfying the inequality*

$$\psi(\rho(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \quad (1.2)$$

for all $x, y \in X$, where M is given by

$$M(x, y) = \max \left\{ \rho(y, Ty) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)}, \rho(x, y) \right\} \quad (1.3)$$

and

$\psi : [0, \infty) \rightarrow [0, \infty)$ is continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Then T has a unique fixed point in X .

Samet et al.[6] introduced the concepts of $\alpha - \psi$ contractive type mappings and α -admissible mappings and proved following results:

Definition 1.6. [6] Let (X, d) be a metric space and $T : X \rightarrow X$ be a self mapping. T is said to be an $\alpha - \psi$ contractive mappings if there exists two functions $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 1.7. [6] Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. T is said to α -admissible if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$$

for all $x, y \in X$.

Theorem 1.8. [6] *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an $\alpha - \psi$ contractive mappings satisfying the conditions:*

- (i) *T is α -admissible;*
- (ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- (iii) *T is continuous.*

Then T has a fixed point.

Theorem 1.9. [6] *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an $\alpha - \psi$ contractive mappings satisfying the conditions:*

- (i) *T is α -admissible;*
- (ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- (iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point. Further, Samet et al.[6] added the condition that for all $x, y \in X$, there exists $u \in X$ such that $\alpha(x, u) \geq 1$ and $\alpha(y, u) \geq 1$ to hypotheses of above theorems to assure the uniqueness of the fixed point.

Definition 1.10. ([1]) Let X be a non-empty set. A function $\rho : X \times X \rightarrow [0, \infty)$ is said to be a partial metric on X if the following conditions hold:

- (i) $x = y$ if and only if $\rho(x, x) = \rho(y, y) = \rho(x, y)$;
- (ii) $\rho(x, x) \leq \rho(x, y)$;
- (iii) $\rho(x, y) = \rho(y, x)$;
- (iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z)$. for all $x, y, z \in X$.

The set X equipped with the metric ρ defined above is called a partial metric space and it is denoted by (X, ρ) (in short PMS). Each partial metric ρ on X generates a T_0 topology τ_ρ on X , which has a base of the family of open ρ -balls

$$\{B_\rho(x, \epsilon) : x \in X, \epsilon > 0\}$$

Where

$$B_\rho(x, \epsilon) = \{y \in X : \rho(x, y) < \rho(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$

Example 1.11. ([12]) Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and define $\rho([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, ρ) is a partial metric space.

Example 1.12. ([12]) Let $X = [0, \infty)$ and define $\rho(x, y) = \max\{x, y\}$. Then (X, ρ) is a partial metric space.

Lemma 1.13. ([1]) *Let (X, ρ) be a partial metric space.*

- (a) *A sequence $\{x_n\}$ in (X, ρ) converges to a point $x \in X$ if and only if*

$$\rho(x, x) = \lim_{n \rightarrow \infty} \rho(x_n, x),$$

- (b) *A sequence $\{x_n\}$ in (X, ρ) is a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \rho(x_n, x_m)$ exists and finite ,*

(c) (X, ρ) is complete if every Cauchy $\{x_n\}$ in X converges to a point $x \in X$, such that

$$\rho(x, x) = \lim_{m, n \rightarrow \infty} \rho(x_m, x_n) = \lim_{n \rightarrow \infty} \rho(x_n, x) = \rho(x, x).$$

Lemma 1.14. ([11], [1], [2]) Let ρ be a partial metric on X , then the functions $d_\rho, d_m : X \times X \rightarrow \mathbb{R}^+$ such that

$$d_\rho(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y)$$

and

$$\begin{aligned} d_m(x, y) &= \max\{\rho(x, y) - \rho(x, x), \rho(x, y) - \rho(y, y)\} \\ &= \rho(x, y) - \min\{\rho(x, x), \rho(y, y)\} \end{aligned}$$

are metric on X . Furthermore (X, d_ρ) and (X, d_m) are metric spaces. It is clear that d_ρ and d_m are equivalent.

Let (X, ρ) be a partial metric space. Then

- (1) A sequence $\{x_n\}$ in (X, ρ) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_ρ) ,
- (2) (X, ρ) is complete if and only if the metric space (X, d_ρ) is complete.

Moreover

$$\lim_{n \rightarrow \infty} d_\rho(x_n, x) = 0 \Leftrightarrow \rho(x, x) = \lim_{n \rightarrow \infty} \rho(x_n, x) = \lim_{n, m \rightarrow \infty} \rho(x_n, x_m).$$

Lemma 1.15. ([10]) Assume that $x_n \rightarrow z$ as $n \rightarrow \infty$ in a partial metric space (X, ρ) such that $\rho(z, z) = 0$. Then $\lim_{n \rightarrow \infty} \rho(x_n, y) = \rho(z, y)$ for every $y \in X$.

Lemma 1.16. [10] If $\{x_n\}$ with $\lim_{n \rightarrow \infty} \rho(x_n, x_{n+1}) = 0$ is not a Cauchy sequence in (X, ρ) , and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$, then following four sequences

$$\begin{aligned} &\rho(x_{m(k)}, x_{n(k)+1}), \rho(x_{m(k)}, x_{n(k)}), \\ &\rho(x_{m(k)-1}, x_{n(k)+1}), \rho(x_{m(k)-1}, x_{n(k)}) \end{aligned}$$

tend to $\mu > 0$ when $k \rightarrow \infty$

Lemma 1.17. ([13]) Let (X, ρ) be a partial metric space.

- (1) if $\rho(x, y) = 0$ then $x = y$,
- (2) If $x \neq y$ then $\rho(x, y) > 0$

2. MAIN RESULTS

Definition 2.1. Let T be a self mapping on X and let $\alpha, \beta : X \times X \rightarrow [0, \infty)$ be a function. We say T is an $\alpha - \beta$ admissible mapping if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ for all $x, y \in X$.

Definition 2.2. Let (X, ρ) be a partial metric space and $T : X \rightarrow X$ be a given self map. We say that T is a almost generalized $\alpha - \beta - \psi - \varphi - \vartheta$ contractive mapping if there exists two functions $\alpha, \beta : X \times X \rightarrow [0, \infty)$ and $\vartheta \in \Theta$, $\psi, \varphi \in \Psi$ and $L \geq 0$ such that for all $x, y \in X$ we have

$$\alpha(x, Tx)\beta(y, Ty)\psi(\rho(Tx, Ty)) \leq \psi(\tilde{\mathcal{M}}_1(x, y)) - \varphi(\tilde{\mathcal{M}}_2(x, y)) + L(\vartheta(\tilde{\mathcal{N}}(x, y))) \quad (2.1)$$

Where

$$\tilde{\mathcal{M}}_1(x, y) = \max \left\{ \rho(x, y), \rho(y, Ty) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)}, \rho(x, Tx) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)} \right\} \quad (2.2)$$

and

$$\tilde{\mathcal{M}}_2(x, y) = \max \{ \rho(x, y), \rho(x, Tx), \rho(y, Ty) \} \quad (2.3)$$

and

$$\tilde{\mathcal{N}}(x, y) = \min \{ d_m(x, Tx), d_m(x, Ty), d_m(y, Tx), d_m(y, Ty) \} \quad (2.4)$$

Definition 2.3. Let (X, ρ) be a complete partial metric space and $T : X \rightarrow X$ and $\alpha, \beta : X \times X \rightarrow [0, \infty)$ be the mappings. We say that X is $(\alpha - \beta)$ regular if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x \in X$, $\alpha(x_n, x_{n+1}) \geq 1$, $\beta(x_n, x_{n+1}) \geq 1$ for all n , there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x_{n(k-1)}) \geq 1$, $\beta(x_{n(k)}, x_{n(k-1)}) \geq 1$ and $\alpha(x, Tx) \geq 1$, $\beta(x, Tx) \geq 1$ for all k .

Theorem 2.4. Let (X, ρ) be a complete partial metric space and $T : X \rightarrow X$ be self mapping. Suppose $\alpha, \beta : X \times X \rightarrow [0, \infty)$ be the mappings satisfying the conditions:

- (i) T is $\alpha - \beta$ admissible;
- (ii) T is almost generalized $\alpha - \beta - \psi - \varphi - \vartheta$ contractive;
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
- (iv) T is continuous;

Then T has a fixed point.

Proof. Let x_0 be an arbitrary point such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Suppose we have a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the existence part of the proof is finished. Suppose $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$. Since T is $\alpha - \beta$ admissible, so

$$\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$$

$$\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1$$

and using induction we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Similarly, we have $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Now, from (2.1) we have

$$\begin{aligned} \psi(\rho(x_n, x_{n+1})) &= \psi(\rho(Tx_{n-1}, Tx_n)) \leq \alpha(x_{n-1}, Tx_{n-1})\beta(x_n, Tx_n)\psi(\rho(Tx_{n-1}, Tx_n)) \\ &\leq \psi(\tilde{\mathcal{M}}_1(x_{n-1}, x_n)) - \varphi(\tilde{\mathcal{M}}_2(x_{n-1}, x_n)) + L(\vartheta(\tilde{\mathcal{N}}(x_{n-1}, x_n))) \end{aligned} \quad (2.5)$$

where

$$\begin{aligned}
\tilde{\mathcal{M}}_1(x_{n-1}, x_n) &= \max \left\{ \rho(x_{n-1}, x_n), \rho(x_n, Tx_{n-1}) \frac{1 + \rho(x_{n-1}, Tx_{n-1})}{1 + \rho(x_{n-1}, x_n)}, \rho(x_{n-1}, Tx_{n-1}) \frac{1 + \rho(x_{n-1}, Tx_{n-1})}{1 + \rho(x_{n-1}, x_n)} \right\} \\
&= \max \left\{ \rho(x_{n-1}, x_n), \rho(x_n, x_{n+1}) \frac{1 + \rho(x_{n-1}, x_n)}{1 + \rho(x_{n-1}, x_n)}, \rho(x_{n-1}, x_n) \frac{1 + \rho(x_{n-1}, x_n)}{1 + \rho(x_{n-1}, x_n)} \right\} \\
&= \max \left\{ \rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n) \right\}
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\tilde{\mathcal{M}}_2(x_{n-1}, x_n) &= \max \{ \rho(x_{n-1}, x_n), \rho(x_{n-1}, Tx_{n-1}), \rho(x_n, Tx_n) \} \\
&= \max \{ \rho(x_{n-1}, x_n), \rho(x_{n-1}, x_n), \rho(x_n, x_{n+1}) \} \\
&= \max \{ \rho(x_{n-1}, x_n), \rho(x_n, x_{n+1}) \}
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\tilde{\mathcal{N}}(x_{n-1}, x_n) &= \min \{ d_m(x_{n-1}, Tx_{n-1}), d_m(x_{n-1}, Tx_n), d_m(x_n, Tx_{n-1}), d_m(x_n, Tx_n) \} \\
&= \min \{ d_m(x_{n-1}, x_n), d_m(x_{n-1}, x_{n+1}), d_m(x_n, x_n), d_m(x_n, x_{n+1}) \} \\
&= 0
\end{aligned} \tag{2.8}$$

From (2.5), (2.6), (2.7), (2.8) and using the fact that $\vartheta(t) = 0$ if and only if $t = 0$, we get that

$$\psi(\rho(x_n, x_{n+1})) \leq \psi(\max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\}) - \varphi(\max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\})$$

Now, if $\rho(x_n, x_{n+1}) > \rho(x_{n-1}, x_n)$ then

$$\psi(\rho(x_n, x_{n+1})) \leq \psi(\rho(x_n, x_{n+1})) - \varphi(\rho(x_n, x_{n+1}))$$

i.e. $\varphi(\rho(x_n, x_{n+1})) = 0$ then $\rho(x_n, x_{n+1}) = 0$ since $\varphi(t) = 0$ if and only if $t = 0$. which is a contradiction since $\rho(x_n, x_{n+1}) > 0$ by lemma 1.17, therefore

$$\rho(x_n, x_{n+1}) \leq \rho(x_{n-1}, x_n) \tag{2.9}$$

and

$$\psi(\rho(x_n, x_{n+1})) \leq \psi(\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)) \tag{2.10}$$

for all n .

Then from (2.9) we get that $\{\rho(x_n, x_{n+1}) : n \in \mathbb{N}\}$ is a decreasing sequence of non negative real numbers. Hence it is convergent to a real number, therefore there exists $r_0 \geq 0$ such that

$$\lim_{n \rightarrow \infty} \rho(x_n, x_{n+1}) = r_0$$

Let $r_0 > 0$. Then taking the limit $n \rightarrow \infty$ in (2.10) and using the definitions of ψ and φ we get

$$\psi(r_0) \leq \psi(r_0) - \varphi(r_0) < \psi(r_0)$$

This is contradiction. Hence

$$\lim_{n \rightarrow \infty} \rho(x_n, x_{n+1}) = 0 \tag{2.11}$$

Now by lemma 1.14, we have

$$\begin{aligned} d_\rho(x_n, x_{n+1}) &= 2\rho(x_n, x_{n+1}) - \rho(x_n, x_n) - \rho(x_{n+1}, x_{n+1}) \\ &= 2\rho(x_n, x_{n+1}) - \min\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \\ &\leq 2\rho(x_n, x_{n+1}) \end{aligned}$$

Therefore by lemma 1.14,

$$\lim_{n \rightarrow \infty} d_\rho(x_n, x_{n+1}) = 0 \quad (2.12)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X i.e. We prove that $\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0$.

We prove it by contradiction.

Let

$$\lim_{n \rightarrow \infty} \rho(x_n, x_m) \neq 0$$

Then sequences in lemma 1.16 tends to $\mu > 0$, when $k \rightarrow \infty$. So we can see that

$$\lim_{k \rightarrow \infty} \rho(x_{m(k)}, x_{n(k)}) = \mu \quad (2.13)$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is smallest integer with $n(k) > m(k) > k$. Then

$$\lim_{k \rightarrow \infty} \rho(x_{n(k)-1}, x_{m(k)}) = \mu \quad (2.14)$$

Again,

$$\rho(x_{m(k)-1}, x_{n(k)-1}) \leq \rho(x_{m(k)-1}, x_{n(k)}) + \rho(x_{n(k)}, x_{n(k)-1}) - \rho(x_{n(k)}, x_{n(k)})$$

Letting $k \rightarrow \infty$ and using lemma 1.16 we get

$$\lim_{k \rightarrow \infty} \rho(x_{m(k)-1}, x_{n(k)-1}) = \mu \quad (2.15)$$

Again note that

$$\begin{aligned} d_m(x_{n(k)-1}, x_{n(k)}) &= \rho(x_{n(k)-1}, x_{n(k)}) - \min\{\rho(x_{n(k)-1}, x_{n(k)-1}), \rho(x_{n(k)}, x_{n(k)})\} \\ &\leq \rho(x_{n(k)-1}, x_{n(k)}) \end{aligned}$$

So, referring to (2.11)

$$\lim_{k \rightarrow \infty} d_m(x_{n(k)-1}, x_{n(k)}) = 0 \quad (2.16)$$

Now in (2.1) replacing x by $x_{n(k)}$ and y by $x_{m(k)}$ respectively, we get

$$\begin{aligned} \psi(\rho(x_{n(k)}, x_{m(k)})) &= \psi(\rho(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \alpha(x_{n(k)-1}, Tx_{n(k)-1})\beta(x_{m(k)-1}, Tx_{m(k)-1})\psi(\rho(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \psi(\tilde{\mathcal{M}}_1(x_{n(k)-1}, x_{m(k)-1})) - \varphi(\tilde{\mathcal{M}}_2(x_{n(k)-1}, x_{m(k)-1})) + L(\vartheta(\tilde{\mathcal{N}}(x_{n(k)-1}, x_{m(k)-1}))) \end{aligned} \quad (2.17)$$

Where

$$\begin{aligned}
\tilde{\mathcal{M}}_1(x_{n(k)-1}, x_{m(k)-1}) &= \max \left\{ \rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{m(k)-1}, Tx_{m(k)-1}) \frac{1 + \rho(x_{n(k)-1}, Tx_{n(k)-1})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \right. \\
&\quad \left. \rho(x_{n(k)-1}, Tx_{n(k)-1}) \frac{1 + \rho(x_{n(k)-1}, Tx_{n(k)-1})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})} \right\} \\
&= \max \left\{ \rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{m(k)-1}, x_{m(k)}) \frac{1 + \rho(x_{n(k)-1}, x_{n(k)})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \right. \\
&\quad \left. \rho(x_{n(k)-1}, x_{n(k)}) \frac{1 + \rho(x_{n(k)-1}, x_{n(k)})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})} \right\} \quad (2.18)
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{M}}_2(x_{n(k)-1}, x_{m(k)-1}) &= \max\{\rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{n(k)-1}, Tx_{n(k)-1}), \rho(x_{m(k)-1}, Tx_{m(k)-1})\} \\
&= \max\{\rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{n(k)-1}, x_{n(k)}), \rho(x_{m(k)-1}, x_{m(k)})\} \quad (2.19)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{N}}(x_{n(k)-1}, x_{m(k)-1}) &= \min\{d_m(x_{n(k)-1}, Tx_{n(k)-1}), d_m(x_{n(k)-1}, Tx_{m(k)-1}), \\
&\quad d_m(x_{m(k)-1}, Tx_{n(k)-1}), d_m(x_{m(k)-1}, Tx_{m(k)-1})\} \\
&= \min\{d_m(x_{n(k)-1}, x_{n(k)}), d_m(x_{n(k)-1}, x_{m(k)}), \\
&\quad d_m(x_{m(k)-1}, x_{n(k)}), d_m(x_{m(k)-1}, x_{m(k)})\} \quad (2.20)
\end{aligned}$$

Letting $k \rightarrow \infty$ in (2.18) and (2.19) and using (2.13), (2.14), (2.15) and lemma 1.16 we get

$$\lim_{k \rightarrow \infty} \tilde{\mathcal{M}}_1(x_{n(k)-1}, x_{m(k)-1}) = \mu \quad (2.21)$$

and

$$\lim_{k \rightarrow \infty} \tilde{\mathcal{M}}_2(x_{n(k)-1}, x_{m(k)-1}) = \mu \quad (2.22)$$

Now again, Letting $k \rightarrow \infty$ in (2.20) and using (2.16) we get

$$\lim_{k \rightarrow \infty} \tilde{\mathcal{N}}(x_{n(k)-1}, x_{m(k)-1}) = 0 \quad (2.23)$$

Now Letting $k \rightarrow \infty$ in (2.17) and using (2.21), (2.21), (2.23) and definition of ϑ we get

$$\psi(\mu) \leq \psi(\mu) - \varphi(\mu) < \psi(\mu)$$

This is a contradiction, Therefore

$$\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0 \quad (2.24)$$

This implies that $\{x_n\}$ is a Cauchy sequence in X . Thus, by lemma 1.14 this sequence will also Cauchy in the metric space (X, d_ρ) . In addition, since (X, ρ) is complete, (X, d_ρ) is also complete. Therefore the sequence $\{x_n\}$ is convergent in the space (X, d_ρ) , say $\lim_{n \rightarrow \infty} d_\rho(x_n, u) = 0$. Again from Lemma 1.13, we get

$$\rho(u, u) = \lim_{n \rightarrow \infty} \rho(x_n, u) = \lim_{m, n \rightarrow \infty} \rho(x_n, x_m) = 0$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the complete partial metric space (X, ρ) , it is convergent to a point x^* in X .
As T is continuous, we have

$$x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Tx_n = Tx^*$$

□

In the following, we omit the continuity assumption of T in Theorem 2.4.

Theorem 2.5. *Let (X, ρ) be a complete partial metric space and $T : X \rightarrow X$ be self mapping. Suppose $\alpha, \beta : X \times X \rightarrow [0, \infty)$ be the mappings satisfying the conditions:*

- (i) T is $\alpha - \beta$ admissible;
- (ii) T is almost generalized $\alpha - \beta - \psi - \varphi - \vartheta$ contractive;
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
- (iv) X is $(\alpha - \beta)$ regular;

Then T has a fixed point. Further if z, z_1 are fixed points of T with $\alpha(z, Tz) \geq 1$, $\alpha(z_1, Tz_1) \geq 1$ and $\beta(z, Tz) \geq 1$, $\beta(z_1, Tz_1) \geq 1$, then $z = z_1$.

Proof. From the proof of the Theorem 2.4, the sequence x_n defined by $x_{n+1} = Tx_n$ is Cauchy in X . Now, suppose that X is $(\alpha - \beta)$ regular. We have to show that $Tx^* = x^*$. Assume that $\rho(Tx^*, x^*) > 0$. Then as X is $(\alpha - \beta)$ regular there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x_{n(k+1)}) \geq 1$, $\beta(x_{n(k)}, x_{n(k+1)}) \geq 1$ and $\alpha(x, Tx) \geq 1$, $\beta(x, Tx) \geq 1$ for all k .

Now in (2.1) replacing x by $x_{n(k)}$ and y by x^* we get

$$\begin{aligned} \psi(\rho(x_{n(k+1)}, Tx^*)) &= \psi(\rho(Tx_{n(k)}, Tx^*)) \\ &\leq \alpha(x_{n(k)}, Tx_{n(k)})\beta(x^*, Tx^*)\psi(\rho(Tx_{n(k)}, Tx^*)) \\ &\leq \psi(\tilde{\mathcal{M}}_1(x_{n(k)}, x^*)) - \varphi(\tilde{\mathcal{M}}_2(x_{n(k)}, x^*)) + L(\vartheta(\tilde{\mathcal{N}}(x_{n(k)}, x^*))) \end{aligned} \quad (2.25)$$

Where

$$\begin{aligned} \tilde{\mathcal{M}}_1(x_{n(k)}, x^*) &= \max \left\{ \rho(x_{n(k)}, x^*), \rho(x^*, Tx^*) \frac{1 + \rho(x_{n(k)}, Tx_{n(k)})}{1 + \rho(x_{n(k)}, x^*)}, \right. \\ &\quad \left. \rho(x_{n(k)}, Tx_{n(k)}) \frac{1 + \rho(x_{n(k)}, Tx_{n(k)})}{1 + \rho(x_{n(k)}, x^*)} \right\} \\ &= \max \left\{ \rho(x_{n(k)}, x^*), \rho(x^*, Tx^*) \frac{1 + \rho(x_{n(k)}, x_{n(k+1)})}{1 + \rho(x_{n(k)}, x^*)}, \right. \\ &\quad \left. \rho(x_{n(k)}, x_{n(k+1)}) \frac{1 + \rho(x_{n(k)}, x_{n(k+1)})}{1 + \rho(x_{n(k)}, x^*)} \right\} \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} \tilde{\mathcal{M}}_2(x_{n(k)}, x^*) &= \max\{\rho(x_{n(k)}, x^*), \rho(x_{n(k)}, Tx_{n(k)}), \rho(x^*, Tx^*)\} \\ &= \max\{\rho(x_{n(k)}, x^*), \rho(x_{n(k)}, x_{n(k+1)}), \rho(x^*, Tx^*)\} \end{aligned} \quad (2.27)$$

and

$$\begin{aligned}\tilde{\mathcal{N}}(x_{n(k)}, x^*) &= \min\{d_m(x_{n(k)}, Tx_{n(k)}), d_m(x_{n(k)}, Tx^*), d_m(x^*, Tx_{n(k)}), d_m(x^*, Tx^*)\} \\ &= \min\{d_m(x_{n(k)}, x_{n(k)+1}), d_m(x_{n(k)}, Tx^*), d_m(x^*, x_{n(k)+1}), d_m(x^*, Tx^*)\}\end{aligned}\quad (2.28)$$

Now, taking $k \rightarrow \infty$ in (2.26), (2.27) and ((2.28) we get

$$\lim_{k \rightarrow \infty} \tilde{\mathcal{M}}_1(x_{n(k)}, x^*) = \rho(x^*, Tx^*) \quad (2.29)$$

and

$$\lim_{k \rightarrow \infty} \tilde{\mathcal{M}}_2(x_{n(k)}, x^*) = \rho(x^*, Tx^*) \quad (2.30)$$

and

$$\lim_{k \rightarrow \infty} \tilde{\mathcal{N}}(x_{n(k)}, x^*) = 0 \quad (2.31)$$

Now, taking $k \rightarrow \infty$ in (2.25) and using (2.29), (2.30), (2.31) and definitions ψ, φ and ϑ we get

$$\psi(\rho(x^*, Tx^*)) \leq \psi(\rho(x^*, Tx^*)) - \varphi(\rho(x^*, Tx^*)) < \psi(\rho(x^*, Tx^*))$$

which is a contradiction. Therefore $Tx^* = x^*$ i.e. x^* is a fixed point.

Further, suppose x^* and y^* be two fixed point of T such that $\alpha(x^*, y^*) \geq 1$, $\beta(x^*, y^*) \geq 1$ and $\rho(x^*, y^*) > 0$ then replacing x by x^* and y by y^* in (2.1) we get

$$\begin{aligned}\psi(\rho(x^*, y^*)) &= \psi(\rho(Tx^*, Ty^*)) \leq \alpha(x^*, Tx^*)\beta(y^*, Ty^*)\rho(Tx^*, Ty^*) \\ &\leq \psi(\tilde{\mathcal{M}}_1(x^*, y^*)) - \varphi(\tilde{\mathcal{M}}_2(x^*, y^*)) + L(\vartheta(\tilde{\mathcal{N}}(x^*, y^*)))\end{aligned}\quad (2.32)$$

Where

$$\begin{aligned}\tilde{\mathcal{M}}_1(x^*, y^*) &= \max\left\{\rho(x^*, y^*), \rho(y^*, Ty^*)\frac{1 + \rho(x^*, Tx^*)}{1 + \rho(x^*, y^*)}, \rho(x^*, Tx^*)\frac{1 + \rho(x^*, Tx^*)}{1 + \rho(x^*, y^*)}\right\} \\ &= \max\left\{\rho(x^*, y^*), \rho(y^*, y^*)\frac{1 + \rho(x^*, x^*)}{1 + \rho(x^*, y^*)}, \rho(x^*, x^*)\frac{1 + \rho(x^*, x^*)}{1 + \rho(x^*, y^*)}\right\} \\ &= \rho(x^*, y^*)\end{aligned}\quad (2.33)$$

and

$$\begin{aligned}\tilde{\mathcal{M}}_2(x^*, y^*) &= \max\{\rho(x^*, y^*), \rho(x^*, Tx^*), \rho(y^*, Ty^*)\} \\ &= \max\{\rho(x^*, y^*), \rho(x^*, x^*), \rho(y^*, y^*)\} \\ &= \rho(x^*, y^*)\end{aligned}\quad (2.34)$$

and

$$\begin{aligned}
\tilde{\mathcal{N}}(x^*, y^*) &= \min\{d_m(x^*, Tx^*), d_m(x^*, Ty^*), d_m(y^*, Tx^*), d_m(y^*, Ty^*)\} \\
&= \min\{d_m(x^*, x^*), d_m(x^*, y^*), d_m(y^*, x^*), d_m(y^*, y^*)\} \\
&= 0
\end{aligned} \tag{2.35}$$

putting (2.33), (2.34), (2.35) in (2.32) and using the fact that $\psi(t) < t$ for $t > 0$ and $\vartheta(t) = 0$ if and only if $t = 0$, we get

$$\psi(\rho(x^*, y^*)) \leq \psi(\rho(x^*, y^*)) - \varphi(\rho(x^*, y^*)) < \psi(\rho(x^*, y^*))$$

which is a contradiction. Hence T has a unique fixed point. This completes the proof. $\square$

Following are consequences of the theorems.

Corollary 2.6. *Let (X, ρ) be a complete partial metric space. $T : X \rightarrow X$ be self-mapping satisfying the condition*

$$\psi(\rho(Tx, Ty)) \leq \psi(\rho(x, y)) - \varphi(\rho(x, y)) + L(\tilde{\mathcal{N}}(x, y)) \tag{2.36}$$

For all $x, y \in X$, $L \geq 0$, $\psi, \varphi \in \Psi$ and

$$\tilde{\mathcal{N}}(x, y) = \min\{d_m(x, Tx), d_m(x, Ty), d_m(y, Tx), d_m(y, Ty)\} \tag{2.37}$$

Then T has a unique fixed point in X .

Corollary 2.7. *Let (X, ρ) be a complete partial metric space. $T : X \rightarrow X$ be self-mapping satisfying the condition*

$$\psi(\rho(Tx, Ty)) \leq \psi(\tilde{\mathcal{M}}(x, y)) - \varphi(\tilde{\mathcal{M}}(x, y)) \tag{2.38}$$

For all $x, y \in X$, $\psi, \varphi \in \Psi$ and

Where

$$\tilde{\mathcal{M}}(x, y) = \max \left\{ \rho(x, y), \rho(y, Ty) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)}, \rho(x, Tx) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)} \right\}$$

Then T has a unique fixed point in X .

Example 2.8. Let $X = [0, \infty)$ and $\rho(x, y) = \max\{x, y\}$. Then (X, ρ) is a complete partial metric space. Consider the mapping $T : X \rightarrow X$ defined by

$$T(x) = \begin{cases} x - \frac{2}{3} & x > 1 \\ \frac{x}{3} & 0 \leq x \leq 1 \\ 0 & x \leq 0 \end{cases} \tag{2.39}$$

and $\psi, \vartheta : [0, \infty) \rightarrow [0, \infty)$ be such that $\psi(t) = t$, $\varphi(t) = \frac{t}{2}$ and $\vartheta(t) = t$ for all $t \geq 0$. If we define the functions $\alpha, \beta : X \times X \rightarrow [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} \frac{3}{2} & x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{2.40}$$

and

$$\beta(x, y) = \begin{cases} 1 & x, y \in [0, 1] \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad (2.41)$$

Without loss of generality we assume that $x \geq y$. Then for $x, y \in [0, 1]$ we get

$$\begin{aligned} \alpha(x, Tx)\beta(y, Ty)\psi(\rho(Tx, Ty)) &= \alpha(x, \frac{x}{3})\beta(y, \frac{y}{3})\psi(\rho(\frac{x}{3}, \frac{y}{3})) \\ &= \frac{3}{2}(\frac{x}{3}) = \frac{x}{2} \end{aligned} \quad (2.42)$$

On the other side

$$\begin{aligned} \tilde{\mathcal{M}}_1(x, y) &= \max \left\{ \rho(x, y), \rho(y, Ty) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)}, \rho(x, Tx) \frac{1 + \rho(x, Tx)}{1 + \rho(x, y)} \right\} \\ &= \max \left\{ \rho(x, y), \rho(y, \frac{y}{3}) \frac{1 + \rho(x, \frac{x}{3})}{1 + \rho(x, y)}, \rho(x, \frac{x}{3}) \frac{1 + \rho(x, \frac{x}{3})}{1 + \rho(x, y)} \right\} \\ &= x \end{aligned} \quad (2.43)$$

and

$$\begin{aligned} \tilde{\mathcal{M}}_2(x, y) &= \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty)\} \\ &= \max\{\rho(x, y), \rho(x, \frac{x}{3}), \rho(y, \frac{y}{3})\} \\ &= x \end{aligned} \quad (2.44)$$

and

$$\begin{aligned} \tilde{\mathcal{N}}(x, y) &= \min\{d_m(x, Tx), d_m(x, Ty), d_m(y, Tx), d_m(y, Ty)\} \\ &= \min\{d_m(x, \frac{x}{3}), d_m(x, \frac{y}{3}), d_m(y, \frac{x}{3}), d_m(y, \frac{y}{3})\} \end{aligned} \quad (2.45)$$

Therefore from (2.43) and (2.44) we get

$$\begin{aligned} \psi(\tilde{\mathcal{M}}_1(x, y)) - \varphi(\tilde{\mathcal{M}}_2(x, y)) + L(\vartheta(\tilde{\mathcal{N}}(x, y))) &= \psi(x) - \varphi(x) + L(\vartheta(\tilde{\mathcal{N}}(x, y))) \\ &= x - \frac{x}{2} + L(\tilde{\mathcal{N}}(x, y)) \\ &= \frac{x}{2} + L(\tilde{\mathcal{N}}(x, y)) \end{aligned} \quad (2.46)$$

Now since $L(\tilde{\mathcal{N}}(x, y)) = L(\min\{d_m(x, \frac{x}{3}), d_m(x, \frac{y}{3}), d_m(y, \frac{x}{3}), d_m(y, \frac{y}{3})\}) \geq 0$ for all $x, y \in X$, and From (2.42) and (2.46) it is clear that it satisfies all the conditions of Theorem 2.4. Hence T has a fixed point, which in this case is 0.

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