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ON THE ENTIRE SOLUTIONS OF COMPLEX NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the nonlinear difference-differential equations and discuss the conception of their entire solutions. Further, we investigate the criteria for existence and nonexistence of the entire solutions and their characteristic on the aspect of value distribution theory.

1. Introduction, Definitions and Notations.

In this article, meromorphic and entire functions are always defined on whole complex plane $\mathbb{C}$. We assume that readers are familiar with standard notations and definitions of Nevanlinna's value distribution theory like as $m(r,f)$, $N(r,\infty;f)$, $T(r,f)$, $S(r,f)$ etc.,(for details, see [3, 11]). The Nevanlinna's value distribution theory has extensive application in the field of complex differential equations and we use the theory as effective tool for Difference-Differential equations in the complex field (for details, see [4]).

Defination 1.1. [3, 11] Deficiency of $\alpha \in \mathbb{C}$ with respect to a meromorphic function $f(z)$ is denoted by $\delta(\alpha,f)$ and defined as

$$\delta(\alpha,f) = \lim_{r\to\infty}\inf \frac{m(r,\frac{1}{f-\alpha})}{T(r,f)} = 1 - \lim_{r\to\infty}\sup \frac{N(r,\alpha;f)}{T(r,f)}.$$

By the Nevanlinna's second fundamental theorem, it can be easily shown that,

$$\sum_{\alpha\in\mathbb{C}\cup\{\infty\}} \delta(\alpha,f) \le 2.$$

Defination 1.2. [3, 11] The order and hyper order of a meromorphic function $f(z)$ are denoted by $\sigma(f)$ and $\sigma_2(f)$, and defined by

$$\sigma(f) = \lim_{r\to\infty}\sup \frac{\log T(r,f)}{\log r},$$

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and

$$\sigma_2(f) = \lim_{r\to\infty} \sup \frac{\log\log T(r,f)}{\log r},$$

respectively. The lower hyper order of $f(z)$ is given by,

$$\nu_2(f) = \lim_{r\to\infty} \inf \frac{\log\log T(r,f)}{\log r}.$$

Using the conception of order and hyper order we ensure the existence and nonexistence of an entire solution of difference-differential equations in the complex field.

Defination 1.3. [3, 11] The exponent of convergence of zeros and poles of a meromorphic function are denoted by $\gamma(f)$ and $\gamma(\frac{1}{f})$ respectively. The exponent and hyper exponent of convergence of zeros of a meromorphic function $f(z)$ are defined by,

$$\gamma(f) = \lim_{r\to\infty} \sup \frac{\log N(r,0;f)}{\log r}.$$

and

$$\gamma_2(f) = \lim_{r\to\infty} \sup \frac{\log\log N(r,0;f)}{\log r}.$$

respectively. $\gamma(\frac{1}{f})$ and $\gamma_2(\frac{1}{f})$ are similarly defined.

Defination 1.4. [3, 11] Let $f(z)$ is function in the complex plane $\mathbb{C}$ and $\omega \in \mathbb{C}\setminus\{0\}$, then $f(z+\omega)$ is call difference or shift function of $f(z)$ and the difference differential equation is given by,

$$Q(z)f^n(z) + A(z)f^{(k)}(z+\omega) = R(z)$$

where $Q(z)$, $A(z)$ are entire function of finite order and $R(z)$ is non-constant polynomial.

Many researchers are focus on existence and solvability of nonlinear difference-differential equation in aspect of Nevanlinna's value distribution theory (see [1, 6, 7, 9]).

In 2011, Zhang and Liao [12] discussed an algebraic difference-differential polynomial $p_d(f)$ and established the following result:

Theorem 1.1. [12] Let algebraic difference-differential polynomial $p_d(f)$ in $f(z)$ of degree $d \leq n-3$, where $n(\geq 4) \in \mathbb{Z}$. If $\mu_1(z)$, $\mu_2(z)$ are nonzero polynomials and ρ_1, ρ_2 arenonzero constants with $\frac{\rho_1}{\rho_2} \neq (\frac{d}{n})^{\pm 1}, 1$, then the equation,

$$f^n(z) + p_d(f) = \mu_1(z)e^{\rho_1 z} + \mu_2(z)e^{-\rho_2 z},$$

does not have any transcendental entire solution of finite order.

In 2015, Xu, Cao and Liu [8], consider the general non-homogeneous difference-differential equation and developed the criteria for nonexistence of it's entire solutions:

Theorem 1.2. [8] Let the nonlinear difference-differential equation,

$$q(z)f^n(z) + a(z)f^{(k)}(z+1) = \mu_1(z)e^{\rho_1(z)} + \mu_2(z)e^{-\rho_2(z)},$$

where $q(z)$, $a(z)$ are two nonzero entire functions of finite order, $\mu_1(z)$, $\mu_2(z)$ are nonzero polynomials and ρ_1, ρ_2 are nonconstant polynomials, $n(\geq 2) \in \mathbb{Z}$. If an entire function $f(z)$ satisfy any one of the following two conditions:
(i) $\gamma(f) < \sigma(f) = \infty, \sigma_2(f) < \infty$;
(ii) $\gamma_2(f) < \sigma_2(f) < \infty$;
then, $f(z)$ can not be a solution of the equation.

On the same paper, they [8] also discussed characteristic of entire solution of a typical difference-differential equation as follow:

Theorem 1.3. [8] Let the nonlinear difference-differential equation,

$$f^3(z) + a(z)f^{(k)}(z+1) = \mu_1 e^{\lambda\rho(z)} + \mu_2 e^{-\lambda\rho(z)},$$

where $a(z)$ be an entire functions of zero order, μ_1, μ_2 and λ are nonzero constants, and $\rho(z)$ nonconstant polynomial. If the equation poses a transcendental entire solution $f(z)$, then $\delta(0, f) = 0$.

Theorem 1.4. [8] Let the nonlinear difference-differential equation,

$$f^3(z) + a(z)f^{(k)}(z+1) = \mu_1 e^{\lambda z} + \mu_2 e^{-\lambda z},$$

where $a(z)$ be an entire functions of zero order, μ_1, μ_2 and λ are nonzero constants. Then the following results are hold,
(i) The equation does not poses any transcendental entire solution of finite order, if $a(z)$ be a nonzero variable;
(ii)The equation poses a transcendental entire solution of finite order, if $a(z)$ be a nonzero constant, and the form of solution is given by,
(a) If k be an even number then the equation poses a transcendental entire solution of finite order if and only if the condition $e^{\frac{\lambda}{3}} = \mp 1$ and $\mu_1\mu_2 = \pm(\frac{a}{3})^3$ holds and if the condition holds, then

$$f(z) = \alpha_j e^{2k\pi iz} - \frac{a}{3\alpha_j}e^{-2k\pi iz},$$

or,

$$f(z) = \alpha_j e^{(2k\pi+\pi)iz} + \frac{a}{3\alpha_j}e^{-(2k\pi+\pi)iz},$$

(b) If k be an odd number then the equation poses a transcendental entire solution of finite order if and only if the condition $e^{\frac{\lambda}{3}} = \mp i$ and $\mu_1\mu_2 = \pm(\frac{ai}{3})^3$ holds and if the condition holds, then

$$f(z) = \alpha_j e^{(2k\pi+\frac{\pi}{2})iz} - \frac{a}{3\alpha_j}e^{-(2k\pi+\frac{\pi}{2})iz},$$

or,

$$f(z) = \alpha_j e^{(2k\pi-\frac{\pi}{2})iz} + \frac{a}{3\alpha_j}e^{-(2k\pi-\frac{\pi}{2})iz}.$$

Defination 1.5. The difference-differential polynomial is defined by $P(z+\omega) = \sum_{j=1}^{k} b_j f^{(j)}(z+\omega)$, where $\omega \in \mathbb{C} \setminus \{0\}$ and $b_j (j = 1, 2, ..., k)$ are constants(atleast one is nonzero).

Considering the difference-differential equation,

$$q(z)f^n(z) + a(z)P(z+\omega) = \mu(z)\sin\lambda\rho(z),$$

we establish the criteria of existence and nonexistence of an entire solution, and find the form of a solution of the equation.

2. Lemmas

In this section, we present some lemmas which will be needed in a sequel.

Lemma 2.1. [11] Let $f_1(z)$, $f_2(z)$,...,$f_n(z)$,$(n \geq 2)$ are meromorphic functions and $g_1(z)$, $g_2(z)$,...,$g_n(z)$, are entire functions satisfying the following conditions:
(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0$;
(ii) $g_j(z) - g_k(z)$ are not constant for $1 \leq j \leq k \leq n$;
(iii) $T(r, f_i) = o(T(r, e^{g_h - g_k}))$, where $1 \leq j \leq n$, $1 \leq h \leq k \leq n$ and $r(\to \infty) \notin E$, where $E \subset (1, \infty)$ is of finite linear measure. Then $f_j \equiv 0$, $j = 1, 2, ..., n$.

Lemma 2.2. [2] Let $f(z)$ be a transcendental entire function with $\sigma(f) = \infty$ and $\sigma_2(f) < \infty$, then $f(z)$ can be represent as $f(z) = \psi(z)e^{\phi(z)}$, where $\psi(z)$ and $\phi(z)$ are entire functions such that $\gamma(\psi) = \sigma(\psi) = \gamma(f)$, $\gamma_2(\psi) = \sigma_2(\psi) = \gamma_2(f)$ and $\sigma_2(f) = \max\{\sigma_2(\psi), \sigma_2(e^{\phi})\}$.

Lemma 2.3. [5] Let $f(z)$ be a transcendental meromorphic solution of finite order of a difference equation of the form,

$$P(z, f)U(z, f) = Q(z, f),$$

where $P(z, f)$, $U(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree of $U(z, f)$ in $f(z)$ and its shifts is n, and that the total degree of $Q(z, f)$ is at most n. If $U(z, f)$ just contains one term of maximal total degree, then for any $\epsilon > 0$,

$$m(r, P(z, f)) = O(r^{\sigma+\epsilon-1}) + S(r, f)$$

holds possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.4. [10] Let the equation,

$$f^2(z) + (cf^{(n)}(z))^2 = \alpha$$

where c is a nonzero constant and α is a nonconstant meromorphic function, then the equation has no transcendental meromorphic solution $f(z)$ satisfying $T(r, \alpha) = S(r, f)$.

3. Main results

Theorem 3.1. Let the nonlinear difference-differential equation,

$$q(z)f^n(z) + a(z)P(z+\omega) = \mu(z)\sin\lambda\rho(z),$$

where $q(z)$, $a(z)$ are two nonzero entire functions of finite order, $\mu(z)$, $\rho(z)$ are nonzero polynomials, and $n(\geq 2) \in \mathbb{Z}$. If an entire function $f(z)$ satisfy any one of the following two conditions:
(i) $\gamma(f) < \sigma(f) = \infty, \sigma_2(f) < \infty$;
(ii) $\gamma_2(f) < \sigma_2(f) < \infty$;
then, $f(z)$ can not be a solution of the equation.

Proof. We assume that $f(z)$ is entire solution of the equation,

$$q(z)f^n(z) + a(z)P(z+\omega) = \mu(z)\sin\lambda\rho(z), \tag{3.1}$$

and $f(z)$ satisfy $\gamma(f) < \sigma(f) = \infty$; $\sigma_2(f) < \infty$.With help of Lemma 2.2, we can write $f(z) = \psi(z)e^{\phi(z)}$, where $\psi(z)$ is an entire function and $\phi(z)$ is a transcendental entire function such that,
$\gamma(\psi) = \sigma(\psi) = \gamma(f)$, $\gamma_2(\psi) = \sigma_2(\psi) = \gamma_2(f)$ and $\sigma_2(f) = \max\{\sigma_2(\psi), \sigma_2(e^\phi)\}$.
We have $\sigma_2(f) < \infty$, then $\sigma(\phi) = \sigma_2(e^\phi) < \infty$. We substitute $f(z) = \psi(z)e^{\phi(z)}$ into (3.1), we have,

$$q(z)\psi^n(z)e^{n\phi(z)} + a(z)\sum_{j=1}^{k}(\psi(z+\omega)e^{\phi(z+w)})^{(j)} = \mu(z)\sin\lambda\rho(z)$$
$$\Rightarrow q(z)\psi^n(z)e^{n\phi(z)} + a(z)\chi(z)e^{\phi(z+w)} = \mu(z)\sin\lambda\rho(z)$$

Where $\chi(z)$ is differential polynomial of $\psi(z+\omega)$ and $\phi(z+\omega)$ and $\sigma(\chi) < \infty$. Again, we take $\varphi(z) = \phi(z+w) - n\phi(z)$. Then we have,

$$q(z)\psi^n(z) + a(z)\chi(z)e^{\varphi(z)} = \mu(z)\sin\lambda\rho(z)e^{-n\phi(z)}$$

Let $\varphi(z)$ is a polynomial, then

$$\sigma(q(z)\psi^n(z) + a(z)\chi(z)e^{\varphi(z)}) < \infty,$$

but,

$$\sigma(\mu(z)\sin\lambda\rho(z)e^{-n\phi(z)}) = \infty,$$

which is contradiction.
Again, if $\varphi(z)$ be a transcendental entire function, then (3.1) can be written as,

$$q(z)\psi^n(z)e^{n\phi(z)} + a(z)\chi(z)e^{\phi(z+w)} - e^{\beta(z)}\mu\sin\lambda\rho(z) = 0,$$

where $\beta(z) \equiv 0$. Then from Lemma 2.1, we deduce that, $q(z)\psi^n(z) \equiv 0$, $a(z)\chi(z) \equiv 0$, $\mu\sin\lambda\rho(z) \equiv 0$. If $q(z)\psi^n(z) \equiv 0$, then $\psi^n(z) \equiv 0$ and hence $f(z) \equiv 0$, which is contradiction as $\sigma(f) = \infty$. Hence, $f(z)$ does not a solution of the equation (3.1).

Again, let $f(z)$ is an entire solution of (3.1) and satisfying $\gamma_2(f) < \sigma_2(f) < \infty$. Using Lemma 2.2, we have $f(z) = \psi(z)e^{\phi(z)}$, where $\psi(z)$ is an entire function and $\phi(z)$ is a transcendental entire function such that,

$\gamma(\psi) = \sigma(\psi) = \gamma(f)$, $\gamma_2(\psi) = \sigma_2(\psi) = \gamma_2(f)$ and $\sigma_2(f) = \max\{\sigma_2(\psi), \sigma_2(e^{\phi})\}$. With the given condition, $\sigma_2(f) = \sigma_2(e^{\phi}) < \infty$, so, $\sigma_2(\psi) < \sigma_2(e^{\phi}) = \sigma(\phi) < \infty$. We substitute $f(z) = \psi(z)e^{\phi(z)}$ into (3.1), we find,

$$q(z)\psi^n(z) + a(z)\chi(z)e^{\varphi(z)} = \mu(z)\sin\lambda\rho(z)e^{-n\phi(z)}$$

Now if $\sigma(\varphi) < \sigma(\phi)$, then,

$$\sigma_2(q(z)\psi^n(z) + a(z)\chi(z)e^{\varphi(z)}) \leq \max\{\sigma_2(\psi), \sigma(\gamma)\} < \sigma(\phi) = \sigma_2(\mu\sin\lambda\rho(z)e^{-n\phi(z)}),$$

which is a contradiction.
Again, if $\sigma(\varphi) = \sigma(\phi)$, then proceeding similarly as above we obtain a contradiction. Hence the theorem. □

Corollary 3.1. One can easily prove the similar result for the nonlinear difference-differential equation

$$q(z)f^n(z) + a(z)P(z+\omega) = \mu(z)\sinh\lambda\rho(z).$$

Remark 3.1. It is easily can be shown that the corollary is a special case of theorem 1.2 for proper choice of $P(z+\omega)$ and μ.

Theorem 3.2. Let the nonlinear difference-differential equation

$$f^3(z) + a(z)P(z+\omega) = \mu\sin\lambda\rho(z),$$

where $a(z)$ be an entire functions of zero order, μ, λ are nonzero constants, and $\rho(z)$ nonconstant polynomial. If the equation poses a transcendental entire solution $f(z)$, then $\delta(0, f) = 0$.

Proof. We assume that $f(z)$ is transcendental entire solution of the equation,

$$f^3(z) + a(z)P(z+\omega) = \mu\sin\lambda\rho(z). \tag{3.2}$$

Differentiating (3.1), we find that,

$$3f^2(z)f'(z) + a'(z)P(z+\omega) + a(z)P'(z+\omega) = \mu\lambda\rho'(z)\cos\lambda\rho(z), \tag{3.3}$$

we find from (3.2) and (3.3) that,

$$(\lambda\rho')^2[f^3(z) + a(z)P(z+\omega)]^2 + [3f^2(z)f'(z) + a'(z)P(z+\omega) + a(z)P']^2 = [\mu\lambda\rho']^2$$

Hence $\phi(z)f^4(z) = \chi(f)$, where $\phi(z) = \lambda\rho'^2 f^2 + 9(f'^2)$ and $\chi(f)$ is a difference-differential polynomial in $f(z)$ with total degree 4. Then we obtain with help of Lemma 2.3 that,

$$T(r,\phi) = m(r,\phi) = O(r^{\sigma+\epsilon-1}) + S(r,f).$$

Hence, ϕ is a small function of $f(z)$. Now we discuss following two cases,

Case I. First we assume that $\phi(z) \equiv 0$. Then,

$$\lambda\rho'^2 f^2 + 9(f'^2) = 0,$$

We find by simple calculation that, $f(z) = ce^{\pm i\frac{1}{3}\lambda\rho(z)}$ where c is a constant. Substituting $f(z)$ in (3.2)we obtain that,

$$(ce^{\pm i\frac{1}{3}\lambda\rho(z)})^3 + a(z)\sum_{j=1}^{k} b_j c(e^{\pm i\frac{1}{3}\lambda\rho(z+\omega)})^{(j)} = \frac{\mu}{2i}(e^{i\lambda\rho(z)} - e^{-i\lambda\rho(z)}).$$

Hence,

$$(c^3-\frac{\mu}{2i})e^{i\lambda\rho(z)}+a(z)\sum_{j=1}^{k}b_jc(e^{i\frac{1}{3}\lambda\rho(z+\omega)})^{(j)}+\frac{\mu}{2i}e^{-i\lambda\rho(z)}=0;$$

or,

$$(c^3+\frac{\mu}{2i})e^{-i\lambda\rho(z)}+a(z)\sum_{j=1}^{k}b_jc(e^{-i\frac{1}{3}\lambda\rho(z+\omega)})^{(j)}-\frac{\mu}{2i}e^{i\lambda\rho(z)}=0;$$

From Lemma 2.1, we find that $\mu=0$, which is a contradiction as μ is a nonzero constant.

Case II. Let $\phi(z)\not\equiv 0$, then,

$$\phi(z)=\lambda\rho'^2f^2+9(f'^2)=f^2\psi(z)$$

Where $\psi(z)=(\lambda\rho'^2+9\frac{(f'^2)}{f^2})$, from the Lemma of Logarithmic Derivative of meromorphic function, we have $m(r,\psi(z))=S(r,f)$. As $\phi(z)\not\equiv 0$, then $\psi(z)\not\equiv 0$. Now we have,

$$\begin{aligned}
& 2T(r,f)=T(r,f^2)=m(r,f^2)=m(r,\frac{\phi(z)}{\psi(z)})\\
\leq\ & m(r,\phi(z))+m(r,\frac{1}{\psi(z)})\\
\leq\ & S(r,f)+m(r,\frac{1}{\psi(z)})\\
\leq\ & S(r,f)+T(r,\frac{1}{\psi(z)})\\
\leq\ & S(r,f)+T(r,\psi(z))\\
\leq\ & S(r,f)+N(r,\infty;\psi(z))\\
\leq\ & S(r,f)+2N(r,0;f).
\end{aligned}$$

Hence,

$$1\leq\frac{S(r,f)}{2T(r,f)}+\frac{N(r,0;f)}{T(r,f)}.$$

Hence,

$$\begin{aligned}
& 1\leq\lim_{r\to\infty}\sup\frac{S(r,f)}{2T(r,f)}+\lim_{r\to\infty}\sup\frac{N(r,0;f)}{T(r,f)}.\\
\Rightarrow\ & 1\leq 1-\delta(0,f).
\end{aligned}$$

Then $\delta(0,f)=0$. Hence, the theorem is completed. □

Corollary 3.2. We also can find the deficient value of zero with respect to an entire solution of the nonlinear difference-differential equation

$$q(z)f^n(z)+a(z)P(z+\omega)=\mu\sinh\lambda\rho(z).$$

Remark 3.2. We can chose $P(z+\omega)$ and μ with proper justification that the corollary will coincide with the theorem 1.3.

We find the form of a solution of the equation of theorem 3.2 by taking $P(z+\omega)=b_k f^{(k)}(z+\omega)$ and $\rho(z)=z$ as follows:

Theorem 3.3. Let the nonlinear difference-differential equation

$$f^3(z)+a(z)b_k f^{(k)}(z+\omega)=\mu\sin\lambda z,$$

where $a(z)$ be an entire functions of zero order, μ and λ are nonzero constants. Then the following results are hold,
(i) The equation does not poses any transcendental entire solution of finite order, if $a(z)$ be a nonzero variable;
(ii)The equation poses a transcendental entire solution of finite order, if $a(z)$ be a nonzero constant, and the form of solution is given by,
(a) If k be an even number then,

$$f(z)=\alpha_j e^{\frac{i}{\omega}(2k\pi)z}-\frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi)z},$$

or,

$$f(z)=\alpha_j e^{\frac{i}{\omega}(2k\pi+\pi)z}+\frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi+\pi)z},$$

(b) If k be an odd number then,

$$f(z)=\alpha_j e^{\frac{i}{\omega}(2k\pi+\frac{\pi}{2})z}-\frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi+\frac{\pi}{2})z},$$

or,

$$f(z)=\alpha_j e^{\frac{i}{\omega}(2k\pi-\frac{\pi}{2})z}+\frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi-\frac{\pi}{2})z},$$

where α_j is cubic root of $\frac{\mu}{2i}$ and $j=1,2,3$.

Proof. We assume that $f(z)$ is transcendental entire solution of the equation,

$$f^3(z)+a(z)b_k f^{(k)}(z+\omega)=\mu\sin\lambda z. \tag{3.4}$$

Differentiating (3.4), we find that,

$$3f^2(z)f'(z)+a'(z)b_k f^{(k)}(z+\omega)+a(z)b_k f^{(k+1)}(z+\omega)=\mu\lambda\cos\lambda z. \tag{3.5}$$

We find from (3.4) and (3.5) that,

$$\begin{aligned}\lambda^2[f^3(z)+a(z)b_k f^{(k)}(z+\omega)]^2 \quad &+ \quad [3f^2(z)f'(z)+a'(z)b_k f^{(k)}(z+\omega)\\ &+ \quad a(z)b_k f^{(k+1)}(z+\omega)]^2=[\mu\lambda]^2\end{aligned}$$

Hence, $\phi(z)f^4(z)=\chi(f)$, where $\phi(z)=\lambda^2 f^2+9(f'^2)$ and $\chi(f)$ is a difference-differential polynomial in $f(z)$ with total degree 4. Then we obtain with help of Lemma 2.3 that,

$$T(r,\phi)=m(r,\phi)=O(r^{\sigma+\epsilon-1})+S(r,f).$$

Hence, ϕ is a small function of $f(z)$. Now we discuss following two cases, Case I. First we assume that $\phi(z) \equiv 0$. Then,

$$\lambda^2 f^2 + 9(f'^2) = 0.$$

We find by simple calculation that, $f(z) = ce^{\pm i\frac{1}{3}\lambda z}$ where c is a constant. Substituting $f(z)$ in (3.4) we obtain that,

$$(ce^{\pm i\frac{1}{3}\lambda z})^3 + a(z)b_k c(e^{\pm i\frac{1}{3}\lambda(z+\omega)})^{(k)} = \frac{\mu}{2i}(e^{i\lambda z} - e^{-i\lambda z}).$$

Hence,

$$(c^3 - \frac{\mu}{2i})e^{i\lambda z} + a(z)b_k c(e^{i\frac{1}{3}\lambda(z+\omega)})^{(k)} + \frac{\mu}{2i}e^{-i\lambda z} = 0;$$

or,

$$(c^3 + \frac{\mu}{2i})e^{-i\lambda z} + a(z)b_k c(e^{i\frac{1}{3}\lambda(z+\omega)})^{(k)} - \frac{\mu}{2i}e^{i\lambda z} = 0;$$

From Lemma 2.1, we find that $\mu = 0$, which is a contradiction as μ is a nonzero constant.

Case II. If $\phi(z) \not\equiv 0$, then by Lemma 2.4, we obtain $\phi(z)$ is a nonzero constant. Then,

$$\phi'^2 2ff' + 18f'f'' = 0.$$

Since $f(z)$ is transcendental, then

$$\lambda^2 f + 9f'' = 0.$$

Solving the differential equation we find that $f(z) = c_1 e^{i\frac{1}{3}\lambda z} + c_2 e^{-i\frac{1}{3}\lambda z}$ where c_1 and c_2 are constants. Hence from (3.4) we can find that,

$$\begin{aligned} &(c_1^3 - \frac{\mu}{2i})e^{i\lambda z} + (c_2^3 + \frac{\mu}{2i})e^{-i\lambda z} + (3c_1^2 c_2 + c_1 a(z)b_k(\frac{1}{3}i\lambda)^k e^{i\frac{1}{3}\lambda\omega})e^{i\frac{1}{3}\lambda z} \\ + \quad &(3c_1c_2^2 + c_2 a(z)b_k(-\frac{1}{3}i\lambda)^k e^{-i\frac{1}{3}\lambda\omega})e^{-i\frac{1}{3}\lambda z} = 0 \end{aligned}$$

Using Lemma 2.1, we obtain that,

$$\begin{aligned} c_1^3 = \frac{\mu}{2i}; c_2^3 = -\frac{\mu}{2i}; 3c_1^2 c_2 + c_1 a(z)b_k(\frac{1}{3}i\lambda)^k e^{i\frac{1}{3}\lambda\omega} = 0; \\ 3c_1c_2^2 + c_2 a(z)b_k(-\frac{1}{3}i\lambda)^k e^{-i\frac{1}{3}\lambda\omega} = 0; \end{aligned} \tag{3.6}$$

From the above result, it is clear that, if $a(z)$ be a non-constant function, then the equation (3.4) has no transcendental entire solution.
If $a(z)$ be a nonzero constant, then the equation poses a transcendental entire solution.

Let k is an even number. Then,

$$\begin{aligned} 3c_1c_2 &= -ab_k(\frac{1}{3}i\lambda)^k e^{i\frac{1}{3}\lambda\omega} = -ab_k(-\frac{1}{3}i\lambda)^k e^{-i\frac{1}{3}\lambda\omega} \\ &\Rightarrow ab_k(\frac{1}{3}i\lambda)^k(e^{i\frac{1}{3}\lambda\omega} - e^{-i\frac{1}{3}\lambda\omega}) = 0 \\ &\Rightarrow e^{i\frac{1}{3}\lambda\omega} = \pm 1. \end{aligned}$$

If $e^{i\frac{1}{3}\lambda\omega} = 1$, then $i\frac{1}{3}\lambda\omega = \log 1 + 2k\pi i$. Hence $i\frac{1}{3}\lambda = \frac{1}{\omega}(2k\pi i)$. or, if $e^{i\frac{1}{3}\lambda\omega} = -1$, then $i\frac{1}{3}\lambda = \frac{1}{\omega}(2k\pi + \pi)i$. Then we have, $c_1c_2 = \mp\frac{1}{3}ab_k(\frac{1}{3}i\lambda)^k$.

Further we assume that c_1 can assume $\alpha_j(j = 1, 2, 3)$ where α_j satisfy the relation, $\alpha_j^3 = \frac{\mu}{2i}$ and then $c_2 = \mp\frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k$ $(j = 1, 2, 3)$. Hence $f(z)$ is of the following form,

$$f(z) = \alpha_j e^{\frac{i}{\omega}(2k\pi)z} - \frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi)z},$$

or,

$$f(z) = \alpha_j e^{\frac{i}{\omega}(2k\pi+\pi)z} + \frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi+\pi)z}.$$

Again, let k is an odd number. Then, from (3.6) we have,

$$\begin{aligned} 3c_1c_2 &= -ab_k(\frac{1}{3}i\lambda)^k e^{i\frac{1}{3}\lambda\omega} = -ab_k(-\frac{1}{3}i\lambda)^k e^{-i\frac{1}{3}\lambda\omega} \\ &\Rightarrow ab_k(\frac{1}{3}i\lambda)^k(e^{i\frac{1}{3}\lambda\omega} + e^{-i\frac{1}{3}\lambda\omega}) = 0 \\ &\Rightarrow e^{i\frac{1}{3}\lambda\omega} = \pm i. \end{aligned}$$

If $e^{i\frac{1}{3}\lambda\omega} = i$, then $i\frac{1}{3}\lambda\omega = \log 1 + (2k\pi + \frac{\pi}{2})i$. Hence $i\frac{1}{3}\lambda = \frac{1}{\omega}(2k\pi + \frac{\pi}{2})i$.
Or, if $e^{i\frac{1}{3}\lambda\omega} = -i$, then $i\frac{1}{3}\lambda = \frac{1}{\omega}(2k\pi - \frac{\pi}{2})i$.
Proceeding similarly, we find,

$$f(z) = \alpha_j e^{\frac{i}{\omega}(2k\pi+\frac{\pi}{2})z} - \frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi+\frac{\pi}{2})z},$$

or,

$$f(z) = \alpha_j e^{\frac{i}{\omega}(2k\pi-\frac{\pi}{2})z} + \frac{1}{3\alpha_j}ab_k(\frac{1}{3}i\lambda)^k e^{-\frac{i}{\omega}(2k\pi-\frac{\pi}{2})z}.$$

Hence, the proof is completed. □

Corollary 3.3. We also can introduce the nonlinear difference-differential equation,

$$q(z)f^n(z) + a(z)b_k f^{(k)}(z + \omega) = \mu \sinh \lambda z,$$

and can find criteria of existence of transcendental entire solution and form of such solution.

Remark 3.3. If we take $b_k = 1$, $\frac{\mu}{2} = \mu_1$ and $\frac{\mu}{-2} = \mu_2$ in the corollary, then the equation will be equivalent with the equation of the theorem 1.4, and hence we get similar type result as theorem 1.4.

Corollary 3.4. We can create η- shift difference-differentiate polynomial as $P(\eta z + \omega) = \sum_{j=1}^{k} b_j f^{(j)}(\eta z + \omega)$, where $\eta, \omega \in \mathbb{C} \setminus \{0\}$ and $b_j (j = 1, 2, ..., k)$ are constants(atleast one is nonzero) and considering $P(\eta z + \omega)$ except $P(z + \omega)$ for the theorem 1.1, theorem 3.2 and theorem 3.3 we can generate similar type results.

4. Open Problems

We can pose following problems from our results,

1. Are the equations (3.1), (3.2) and (3.4) pose transcendental meromorphic solution?
2. Can the equation (3.2) be solved for higher power of $f(z)$?
3. May be omitted the condition $\sigma_2(f) < \infty$ from the theorem 3.1?

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