Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 11 (2023) 44-53

REMARKS ON MULTIPLICATIVE GENERALIZED $(\alpha,\alpha)-$ DERIVATIONS IN IDEALS OF PRIME RINGS

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Abstract. Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and $(F,f),(G,g)$ and (H,h) are multiplicative generalized $(\alpha,\alpha)-$ derivations of R. In this work, we investigate the behaviour of certain differantial identities involving three multiplicative generalized $(\alpha,\alpha)-$ derivations on ideals of prime rings.

1. Introduction

Let R be an associative ring with center Z. Recall that a ring R is prime if $xRy=0$ implies $x=0$ or $y=0$. For any $x,y\in R$ the symbol $[x,y]$ represents the Lie commutator $xy-yx$ and the Jordan product $xoy=xy+yx$. We shall frequently use the following basic commutator identities in the proof of our results:

i) $[x,yz]=y[x,z]+[x,y]z$
ii) $[xy,z]=[x,z]y+x[y,z]$
iii) $xyoz=(xoz)y+x[y,z]=x(yoz)-[x,z]y$
iv) $xoyz=y(xoz)+[x,y]z=(xoy)z-y[z,x]$
v) $[xy,z]_{\alpha,\alpha}=x\,[y,z]_{\alpha,\alpha}+[x,\alpha(z)]\,y=x[y,\alpha(z)]+[x,z]_{\alpha,\alpha}\,y$
vi) $[x,yz]_{\alpha,\alpha}=\alpha(y)\,[x,z]_{\alpha,\alpha}+[x,y]_{\alpha,\alpha}\,\alpha(z)$
vii) $(xz\circ y)_{\alpha,\alpha}=x(z\circ y)_{\alpha,\alpha}-[x,\alpha(y)]z$.

The commutativity of prime rings with derivation was initiated by Posner [13]. An additive map $d:R\to R$ is called a derivation if $d(xy)=d(x)y+xd(y)$, for all $x,y\in R$. A derivation d is said to be inner if there exists $a\in R$ such that $d(x)=[a,x]$, for all $x\in R$. Over the last several years, a number of authors studied commutativity theorems for prime rings admitting automorphisms or derivations on appropriate subsets of R. In [3], Bresar introduced it definition of generalized derivation as follows:

$F:R\to R$ is an additive mapping which is uniquely determined by a derivation d such that $F(xy)=F(x)y+xd(y)$, for all $x,y\in R$. Obviously, every derivation is a generalized derivation. Generalized derivations have been primarily studied on operator algebras.

Date: Received: Nov 23, 2022; Accepted: Jan 13, 2023.
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2010 *Mathematics Subject Classification.* Primary 16W25; Secondary 16N60.
Key words and phrases. Prime ring, ideal, derivation, generalized $(\alpha,\alpha)-$ derivation, multiplicative generalized $(\alpha,\alpha)-$ derivation.

In [4], the concept of multiplicative derivation was defined by Daif, put forward by Martindale in [10], $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y+xd(y)$, holds for all $x, y \in R$. These maps are not additive. In [9], Goldman and Semrl gave the complete description of these maps. We have $R = \mathbb{C}[0, 1]$, the ring of all continuous (real or complex valued) functions and define a mapping $d : R \to R$ such as

$$d(f)(x) = \left\{ \begin{array}{ll} f(x)\log|f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear that d is multiplicative derivation, but d is not additive. Inspired by the definition multiplicative derivation, the notion of multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [6]. $F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and Ali gave a slight generalization of this definition taking d is any mapping (not necessarily an additive mapping or a derivation) in [7]. Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, generalized derivations and the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. Every generalized derivation is a multiplicative generalized derivation. But the converse is not ture in general. The following example justifies this:

Example 1.1. [7, Example 1.1]Let S be any ring and

$$R = \left\{ \left(\begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right) \mid a, b, c \in S \right\}.$$

Define the maps d and $F : R \to R$ as follows:

$$d\left(\begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \text{ ve } F\left(\begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Then it is straightforward to verify that F is a multiplicative generalized derivation associated with a multiplicative derivation d, but F is not a generalized derivation of R.

In [1] Ashraf and Rehman proved that if R is a prime ring with a nonzero ideal I of R and d is a derivation of R such that either $d(xy)-xy \in Z$ or $d(xy)-xy \in Z$, for all $x, y \in I$, then R is commutative. Ashraf et al. obtained that a prime ring R must be commutative, if R satisfies any one of the following conditions: i) $F(xy) \pm xy \in Z$, ii) $F(xy) \pm yx \in Z$, iii) $F(x)F(y) \pm xy \in Z$, for all $x, y \in I$, where F is a generalized derivation of R associated with derivation d and I is a nonzero two sided ideal of R in [2]. Tiwari et al. proved that a prime ring R must be commutative if it admits generalized derivations (F, f) and (G, g) satisfying $F(x)F(y)\pm G(yx)\pm xy \in Z$ or $F(x)F(y)\pm G(xy)\pm xy \in Z$, for all $x, y \in I$, where I is a nonzero two sided ideal of R in [12]. Further the authors have studied the situations: if R admits generalized derivations (F, f), (G, g) and (H, h) satisfying $F(x)G(y) \pm H(yx) \in Z$ or $F(x)G(y) \pm H(xy) \in Z$ for all $x, y \in I$, where I is

a nonzero two sided ideal of R in [8]. Recently, Rehman et al. discussed the behaviour of certain differantial identities involving three generalized derivation on ideals of prime rings in [14].

On the other hand, Daif and Bell proved that R is semiprime ring, I is a nonzero ideal of R and d is a derivation of R such that $d([x,y]) = \pm[x,y]$, for all $x, y \in I$, then R contains a nonzero central ideal in [5]. Many authors generalized this result by taking different derivations on appropriate subsets of R or investigated the same situations in the center of the ring R. Malleswari et al. proved some commutativity theorems for a ring R with a generalized derivation satisfying specific differential identities on ideals of semiprime ring R in [11].

In this article, we generalize the concept of a multiplicative generalized derivation to a multiplicative generalized $(\alpha,\alpha)-$derivation. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative $(\alpha,\alpha)-$ derivation if there exists a map $\alpha : R \to R$ such that $f(xy) = f(x)\alpha(y) + \alpha(x)f(y)$, for all $x, y \in R$. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative generalized $(\alpha,\alpha)-$ derivation if $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$, for all $x, y \in R$ where d the multiplicative $(\alpha,\alpha)-$ derivation of R. Of course a multiplicative generalized $(1,1)-$derivation where 1 is the identity map on R is a multiplicative generalized derivation. Our aim is to prove that ring R is commutative with the help of (F,f), (G,g) and (H,h) are a multiplicative generalized $(\alpha,\alpha)-$ derivations on ideals in prime rings with above conditions. Also we generalized some these results by taking the same situations in the center.

2. Results

Lemma 2.1. [4, Lemma 2]*(a) If R is a semiprime ring, then the center of a nonzero one-sided ideal is contained in the center of R. In particular, any commutative one-sided ideal is contained in the center of R.*

(b) If R is prime with a nonzero central ideal, then R must be commutative.

Lemma 2.2. *Let R be a prime ring and I a nonzero ideal of R. If $I \circ I \subset Z$, then R is a commutative ring.*

Proof. We get

$$x \circ y \in Z, \text{ for all } x, y \in I.$$

Replacing y by yx in the last expression, we obtain that

$$(x \circ y)\, x \in Z, \text{ for all } x, y \in I.$$

This implies that

$$[(x \circ y)\, x, r] = 0, \text{ for all } x, y \in I,$$

and so

$$(x \circ y)[x, r] = 0, \text{ for all } x, y \in I, r \in R.$$

Replacing y by $ys, s \in R$ in the above equation and using this, we get

$$[x, y]s[x, r] = 0, \text{ for all } x, y \in I, r, s \in R.$$

Replacing r by y in this equation, we have

$$[x, y]R[x, y] = (0), \text{ for all } x, y \in I.$$

Since R is prime ring, we get

$$[x,y]=0, \text{ for all } x,y\in I.$$

By Lemma 1, we conclude that R is commutative ring. This comlpletes the proof. □

Theorem 2.3. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f), (G,g) and (H,h) are a multiplicative generalized $(\alpha,\alpha)-$derivations such that $h(Z)\neq 0$. If $F(x)G(y)+H(xy)\pm\alpha(yx)\in Z$ or $F(x)G(y)+H(xy)\pm\alpha(xy)\in Z$, for all $x,y\in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(x)G(y)+H(xy)+\alpha(yx)\in Z, \text{ for all } x,y\in I. \tag{2.1}$$

Replacing yz by y in (2.1), where $z\in Z$, we get

$$F(x)G(y)\alpha(z)+F(x)\alpha(y)g(z)+H(xy)\alpha(z)+\alpha(xy)h(z)+\alpha(yx)\alpha(z)\in Z.$$

Using $\alpha(z)\in Z$ and (2.1) in this expression, we have

$$F(x)\alpha(y)g(z)+\alpha(xy)h(z)\in Z, \text{ for all } x,y\in I. \tag{2.2}$$

Taking xz instead of x in (2.2) and using $f(z)\in Z$, we obtain that

$$F(x)\alpha(zy)g(z)+\alpha(xy)f(z)g(z)+\alpha(xyz)h(z)\in Z, \text{ for all } x,y\in I. \tag{2.3}$$

Writting yz by y in (2.2), we get

$$F(x)\alpha(yz)g(z)+\alpha(xyz)h(z)\in Z, \text{ for all } x,y\in I. \tag{2.4}$$

Subtracting (2.4) from (2.3), we arrive at

$$\alpha(xy)f(z)g(z)\in Z, \quad \text{for all } x,y\in I, z\in Z. \tag{2.5}$$

Since R is prime ring and $f(z),g(z)\in Z$, we have

$$f(z)g(z)=0 \text{ or } \alpha(xy)\in Z, \text{ for all } x,y\in I.$$

If $f(z)g(z)\neq 0$, then we get $\alpha(xy)\in Z$. Using α is an automorphism of R, we obtain that

$$xy\in Z, \text{ for all } x,y\in I.$$

Hence we have $0=[xy,x]=x[y,x]$, for all $x,y\in I$. Replacing yr by y where $r\in R$ in this equation and using this, we arrive at

$$xy[r,x]=0, \text{ for all } x,y\in I, r\in R.$$

By the primeness of R yields that $x=0$ or $x\in Z$, and so $x\in Z$, for all $x\in I$. That is R is commutative, by Lemma 1.

Now, we assume $f(z)g(z)=0$. Since $f(z),g(z)\in Z$, then we have $f(z)=0$ or $g(z)=0$, for all $z\in Z$.

Let $L=\{z\in Z\mid f(z)=0\}$ and $K=\{z\in Z\mid g(z)=0\}$. Clearly each of L and K is additive subgroup of Z such that $Z=L\cup K$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $L=Z$ or $K=Z$.

Assume that $L = Z$. Hence we get $f(Z) = 0$. Replacing x with xz in (2.1), we have

$$F(xz)G(y) + H(xzy) + \alpha(yxz) \in Z$$
$$F(x)\alpha(z)G(y) + \alpha(x)f(z)G(y) + H(xyz) + \alpha(yxz) \in Z.$$

Using $f(z) = 0$ and $\alpha(z) \in Z$, we get

$$F(x)G(y)\alpha(z) + H(xy)\alpha(z) + \alpha(xy)h(z) + \alpha(yx)\alpha(z) \in Z$$

and so

$$(F(x)G(y) + H(xy) + \alpha(yx))\alpha(z) + \alpha(xy)h(z) \in Z, \text{ for all } x, y \in I, z \in Z.$$

Using (2.1) and $\alpha(z) \in Z$ in this expression, we find that

$$\alpha(xy)h(z) \in Z, \text{ for all } x, y \in I, z \in Z. \tag{2.6}$$

Since R is prime ring and $0 \neq h(z) \in Z$, then we have $\alpha(xy) \in Z$. That is $xy \in Z$, for all $x, y \in I$. Using the same arguments as used above, we obtain that $I \subseteq Z$, and so R is commutative by Lemma 1.

Now, we suppose that $K = Z$, and so $g(Z) = 0$. From the equation (2.2), we get

$$\alpha(xy)h(z) \in Z \text{ for all } x, y \in I, \text{ where } z \in Z.$$

Since R is prime ring and $0 \neq h(z) \in Z$, we have $\alpha(xy) \in Z$, and so, $xy \in Z$, for all $x, y \in I$. Using the same techniques as above, we find that $I \subseteq Z$. Thus we conclude that R is commutative by Lemma 1. The proof is completed.

By similar manner, the same conclusion holds for $F(x)G(y)+H(xy)-\alpha(yx) \in Z$, for all $x, y \in I$. Also, we can prove the condition $F(x)G(y)+H(xy)\pm\alpha(xy) \in Z$, for all $x, y \in I$ using the same techniques as used this proof. □

Theorem 2.4. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F, f), (G, g) and (H, h) are a multiplicative generalized $(\alpha, \alpha)-$derivations such that $h(Z) \neq 0$. If $F(x)G(y) \pm H(yx) \in Z$ or $F(x)G(y) \pm H(xy) \in Z$, for all $x, y \in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(x)G(y) + H(yx) \in Z, \text{ for all } x, y \in I. \tag{2.7}$$

Replacing x by $xz, z \in Z$ in (2.7) for all $z \in I$, we obtain that

$$F(x)\alpha(z)G(y) + \alpha(x)f(z)G(y) + H(yx)\alpha(z) + \alpha(yx)h(z) \in Z.$$

Using $\alpha(z) \in Z$, we get

$$(F(x)G(y) + H(yx))\alpha(z) + \alpha(x)f(z)G(y) + \alpha(yx)h(z) \in Z$$

and so

$$\alpha(x)f(z)G(y) + \alpha(yx)h(z) \in Z, \text{ for all } x, y \in I, z \in Z. \tag{2.8}$$

Writing y by $yz, z \in Z$ in (2.8), we have

$$\alpha(x)f(z)G(y)\alpha(z) + \alpha(x)f(z)\alpha(y)g(z) + \alpha(yzx)h(z) \in Z.$$

Again using $\alpha(z) \in Z$ and (2.8), we get

$$\alpha(xy)f(z)g(z) \in Z, \text{ for all } x, y \in I, z \in Z.$$

Since R is prime ring and $f(z), g(z) \in Z$, we find that

$$f(z)g(z) = 0 \text{ or } \alpha(xy) \in Z, \text{ for all } x, y \in I.$$

If $f(z)g(z) \neq 0$, then we get $\alpha(xy) \in Z$, and so, we obtain that $xy \in Z$, for all $x, y \in I$. Using the same methods in the proof of Theorem 1, we find that R is commutative.

Now, we assume $f(z)g(z) = 0$. Since $f(z), g(z) \in Z$, then we have $f(z) = 0$ or $g(z) = 0$, for all $z \in Z$.

Let $L = \{z \in Z \mid f(z) = 0\}$ and $K = \{z \in Z \mid g(z) = 0\}$. Using Baruer's Trick we have $L = Z$ or $K = Z$.

Assume that $L = Z$. Hence we get $f(Z) = 0$. Replacing x with xz in hypothesis, we have

$$F(x)\alpha(z)G(y) + \alpha(x)f(z)G(y) + H(yx)\alpha(z) + \alpha(y)h(z)\alpha(x) \in Z.$$

Using $f(z) = 0$ and $\alpha(z), h(z) \in Z$, we get

$$(F(x)G(y) + H(yx))\alpha(z) + \alpha(yx)h(z) \in Z, \text{ for all } x, y \in I, z \in Z.$$

Again using (2.7) and $\alpha(z) \in Z$ in this expression, we find that

$$\alpha(yx)h(z) \in Z, \text{ for all } x, y \in I, z \in Z.$$

Since R is prime ring and $0 \neq h(z) \in Z$, we have $\alpha(yx) \in Z$, and so, $yx \in Z$, for all $x, y \in I$. Using the same techniques as above, we find that $I \subseteq Z$. Thus we conclude that R is commutative by Lemma 1. The proof is completed.

Now, we suppose that $K = Z$, and so $g(Z) = 0$. Replacing y by $yz, z \in Z$ in the hypothesis, we obtain that

$$F(x)G(y)\alpha(z) + F(x)\alpha(y)g(z) + H(yx)\alpha(z) + \alpha(yx)h(z) \in Z.$$

Using $\alpha(z) \in Z$ and $g(z) = 0$, we get

$$(F(x)G(y) + H(yx))\alpha(z) + \alpha(yx)h(z) \in Z$$

and so

$$\alpha(yx)h(z) \in Z, \text{ for all } x, y \in I, z \in Z. \tag{2.9}$$

Since R is prime ring and $0 \neq h(z) \in Z$, we have $\alpha(yx) \in Z$, and so, $yx \in Z$, for all $x, y \in I$. Using the similar methods as above, we find that $I \subseteq Z$, and so R is commutative by Lemma 1. The proof is completed.

By similar manner, the same conclusion holds for $F(x)G(y)+H(xy)\pm\alpha(yx) \in Z$, for all $x, y \in I$. Also, we can prove the condition $F(x)G(y)+H(xy)\pm\alpha(xy) \in Z$, for all $x, y \in I$ using the same techniques as used this proof.

The same result holds for $F(x)G(y) - H(yx) \in Z$, for all $x, y \in I$.

By similar manner, we get the required result the statuation $F(x)G(y) \pm H(xy) \in Z$, for all $x, y \in I$. □

Theorem 2.5. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F, f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f \neq 0$ and H be an $\alpha-$ multiplicative left multiplier of R. If $F(xoy) \pm H(xoy) = 0$, for all $x, y \in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(xoy) \pm H(xoy) = 0, \text{ for all } x, y \in I. \tag{2.10}$$

Replacing y by yx in (2.10) and using the hypothesis, we get

$$\begin{aligned} 0 &= F(xoyx) \pm H(xoyx) = F((xoy)x) \pm H((xoy)x) \\ &= F(xoy)\alpha(x) + \alpha(xoy) f(x) \pm H(xoy)\alpha(x) \\ &= (F(xoy) \pm H(xoy))\alpha(x) + \alpha(xoy) f(x) \end{aligned}$$

and so

$$\alpha(xoy) f(x) = 0, \text{ for all } x, y \in I \tag{2.11}$$

Writing y by $yt, t \in I$ in the last equation and using this, we find that

$$\alpha([x, y])\alpha(t) f(x) = 0$$

and so

$$\alpha([x, y])V f(x) = (0), \text{ for all } x, y \in I.$$

where $\alpha(I) = V$ is a nonzero ideal of R. Since R is prime and $V \neq (0)$, we find that $\alpha([x, y]) = 0$ or $f(x) = 0$, for each $x \in I, y \in I$.

Let $K = \{x \in I \mid f(x) = 0\}$ and $L = \{x \in I \mid [x, y] = 0, \text{ for all } y \in I\}$. Clearly each of K and L is additive subgroup of I such that $I = K \cup L$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $K = I$ or $L = I$. In the former case, $f(I) = (0)$. Hence we get $f(xr) = 0$, for all $x \in I, r \in R$, and so $f(x)\alpha(r) + \alpha(x)f(r) = 0$. Using $f(x) = 0$, we find that $\alpha(x)f(r) = 0$, for all $x \in I, r \in R$. Since I is a nonzero ideal of prime ring R, we find that $f = 0$ which forces our hypothesis. So we must have $L = I$. Hence $[x, y] = 0$, for all $x, y \in I$. Writing x by $xr, x \in I, r \in R$ in this equation and using this, we obtained that $x[r, y] = 0$, for all $x, y \in I, r \in R$, and so $I \subseteq Z$. Hence we have R is commutative by Lemma 1. The proof is completed. □

Theorem 2.6. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F, f) a multiplicative generalized $(\alpha, \alpha)-$derivation such that $f \neq 0, f(Z) \neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F(xoy) \pm H(xoy) \in Z$, for all $x, y \in I$, then R is commutative.*

Proof. Replacing y by $yz, z \in Z$ in the hypothesis, we get

$$(F(xoy) \pm H(xoy))\alpha(z) + \alpha(xoy) f(z) \in Z, \text{ for all } x, y \in I.$$

Using the hypothesis and $\alpha(z) \in Z$, we have

$$\alpha(xoy) f(z) \in Z, \text{ for all } x, y \in I. \tag{2.12}$$

Since R is prime ring and $f(z) \in Z$, we find that

$$\alpha(xoy) \in Z \text{ or } f(z) = 0, \text{ for all } x, y \in I, z \in Z. \tag{2.13}$$

Using $f(Z) \neq 0$, we get $\alpha(xoy) \in Z$, for all $x, y \in I$. That is $I \circ I \subset Z$, and so R is a commutative ring by Lemma 2. □

Theorem 2.7. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f\neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F(xoy)\pm H([x,y])=0$, for all $x,y\in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(xoy)\pm H([x,y])=0,\ \text{for all } x,y\in I.$$

Replacing y by yx in this equation and using the hypothesis, we get

$$\begin{aligned}0&=F(xoyx)\pm H([x,yx])=F((xoy)x)\pm H([x,y]x)\\&=F(xoy)\alpha(x)\pm\alpha\left(xoy\right)f(x)\pm H([x,y])\alpha(x)\\&=(F(xoy)\pm H([x,y]))\alpha(x)\pm\alpha\left(xoy\right)f(x)\end{aligned}$$

and so

$$\alpha\left(xoy\right)f(x)=0,\ \text{for all}\ \ x,y\in I.$$

This equation same as (2.11) in the proof of Theorem 3. Using the same arguments the proof of Theorem 3, we get the required result. □

Theorem 2.8. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f\neq 0, f(Z)\neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F(xoy)\pm H([x,y])\in Z$, for all $x,y\in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(xoy)\pm H([x,y])\in Z,\ \text{for all } x,y\in I.$$

Replacing y by $yz, z\in Z$ in this expression and using the hypothesis, $\alpha\left(z\right)\in Z$, we get

$$\alpha\left(xoy\right)f(z)\in Z,\ \text{for all}\ \ x,y\in I,z\in Z.$$

The rest of the proof is the same as equation (2.12). Hence we arrive at R is commutative. This completes the proof. □

Theorem 2.9. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f\neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F(xoy)\pm[\alpha(x),H(y)]=0$, for all $x,y\in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(xoy)\pm[\alpha(x),H(y)]=0,\ \text{for all } x,y\in I.$$

Replacing y by yx in this equation and using the hypothesis, we get

$$\begin{aligned}0&=F(xoyx)\pm[\alpha(x),H(yx)]=F((xoy)x)\pm[\alpha(x),H(y)\alpha(x)]\\&=F(xoy)\alpha(x)\pm\alpha\left(xoy\right)f(x)\pm[\alpha(x),H(y)]\alpha(x)\\&=(F(xoy)\pm[\alpha(x),H(y)])\alpha(x)\pm\alpha\left(xoy\right)f(x)\end{aligned}$$

and so

$$\alpha\left(xoy\right)f(x)=0,\ \text{for all}\ \ x,y\in I.$$

This equation same as (2.11) in the proof of Theorem 3. Using the same arguments the proof of Theorem 3, we get the required result. □

Theorem 2.10. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f\neq 0, f(Z)\neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F(xoy)\pm[\alpha(x),H(y)]\in Z$, for all $x,y\in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$F(xoy)\pm[\alpha(x),H(y)]\in Z,\ \text{for all } x,y\in I.$$

Replacing y by $yz, z\in Z$ in this expression and using the hypothesis, $\alpha(z)\in Z$, we get

$$\alpha(xoy)f(z)\in Z,\ \text{for all}\ \ x,y\in I, z\in Z.$$

Using the same techniques after equation (2.12) in the proof of Theorem 4, we arrive at R is commutative, and so, we complete the proof. □

Theorem 2.11. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f\neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F([x,y])\pm[\alpha(x),H(y)]=0$, for all $x,y\in I$, then R is commutative.*

Proof. By the hypothesis, we get

$$F([x,y])\pm[\alpha(x),H(y)]=0,\ \text{for all } x,y\in I.$$

Replacing y by yx in this equation and using the hypothesis, we have

$$\begin{aligned}0&=F([x,yx])\pm[\alpha(x),H(yx)]=F([x,y]x)\pm[\alpha(x),H(y)\alpha(x)]\\&=F([x,y])\alpha(x)\pm\alpha([x,y])f(x)\pm[\alpha(x),H(y)]\alpha(x)\\&=(F([x,y])\pm[\alpha(x),H(y)])\alpha(x)\pm\alpha([x,y])f(x)\end{aligned}$$

and so

$$\alpha([x,y])f(x)=0,\ \text{for all}\ \ x,y\in I.$$

Using the similar arguments after the equation (2.11) in the proof of Theorem 3, we get the required result. □

Theorem 2.12. *Let R be a prime ring, I a nonzero ideal of R, α an otomorphism on R and (F,f) a multiplicative generalized $(\alpha,\alpha)-$derivation such that $f\neq 0, f(Z)\neq 0$ and H be a $\alpha-$ multiplicative left multiplier of R. If $F([x,y])\pm[\alpha(x),H(y)]\in Z$, for all $x,y\in I$, then R is commutative.*

Proof. From our hypothesis, we get

$$F([x,y])\pm[\alpha(x),H(y)]\in Z,\ \text{for all } x,y\in I.$$

Replacing y by $yz, z\in Z$ in this expression and using the hypothesis, $\alpha(z)\in Z$, we have

$$\alpha([x,y])f(z)\in Z,\ \text{for all}\ \ x,y\in I, z\in Z.$$

Using the same arguments after equation (2.12), we obtain the required result. □

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