Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 11 (2023) 29-43

DEGREE k^{th} POWER SUM POLYNOMIAL OF GENERALIZED GRAPHS RELATED TO THE CYCLE

[1*]B. BASAVANAGOUD, [2]SHRUTI POLICEPATIL AND [3]POOJA B.

ABSTRACT. The purpose of this paper is to introduce the concept of degree k^{th} power sum matrix of a graph G. There are some flaws in the definitions of generalized graphs related to the cycle and we correct these definitions. We compute degree k^{th} power sum polynomials of generalized graphs related to cycle. We establish the bounds for the largest degree k^{th} power sum eigenvalue and degree k^{th} power sum energy of graphs.

1. INTRODUCTION AND PRELIMINARIES

Graph energy is described by Gutman [12] as the sum of absolute values of the eigenvalues of the associated adjacency matrix of a graph G, which is related to the total π-electron energy in a molecular graph. Several graph polynomials based on different matrices defined on the graph, such as adjacency matrix [9], Laplacian matrix [16], signless Laplacian matrix [10, 18], distance matrix [1], degree sum matrix [15, 19], are found in the literature of graph theory. For further information refer to [2, 3, 6, 7, 14] .

Let $G = (n, m)$ be a simple, connected and undirected graph. Let G's vertex and edge sets be $V(G)$ and $E(G)$, respectively. The number of edges incident to a vertex v in $V(G)$ is the *degree* $deg_G(v)$ (or $d_G(v)$). Let $d_i = deg_G(v_i)$. If the degree of each vertex in G is r, the graph is r-regular.

The adjacency matrix $A(G)$ of a graph G is a square matrix of order n whose (i, j)-entry is equal to unity if the vertex v_i is adjacent to v_j, and is equal to zero otherwise. The concept of degree sum matrix $DS(G)$ introduced by Ramane et al. [19]. The degree sum matrix of a graph [19] G is an $n \times n$ matrix $DS(G) = [d_{ij}]$, whose elements are defined as

$$d_{ij} = \begin{cases} d_i + d_j, & if\ i \neq j \\ 0, & otherwise. \end{cases}$$

Basavanagoud et al. [2] introduced the concept of degree square sum matrix $DSS(G)$. The degree square sum matrix of a graph [2] G is an $n \times n$ matrix

Date: Received: Nov 19, 2022; Accepted: Jan 1, 2023.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 05C07; Secondary 05C50.

Key words and phrases. degree k^{th} power sum matrix, degree k^{th} power sum polynomial, degree k^{th} power sum eigenvalues, degree k^{th} power sum energy.

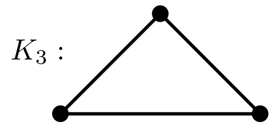


FIGURE 1. Graph K_3.

$DSS(G) = [d_{ij}]$, whose elements are defined as

$$dss_{ij} = \begin{cases} d_i^2 + d_j^2, & if\ i \neq j \\ 0, & otherwise. \end{cases}$$

Motivated by degree sum matrix [19] and degree square sum matrix [2], we introduce the degree k^{th} power sum matrix $D^kS(G)$ of a graph G. Let G be a simple graph with n vertices $v_1, v_2, ..., v_n$ and let d_i be the degree of v_i, $i = 1, 2, ..., n$. Then $D^kS(G) = [d^k s_{ij}]$ for any integer $k \geq 1$, is called the degree k^{th} power sum matrix of a graph G, where

$$d^k s_{ij} = \begin{cases} d_i^k + d_j^k, & if\ i \neq j \\ 0, & otherwise. \end{cases}$$

Some of hitherto studied matrices defined on graph are the special case of degree k^{th} power sum matrix, for particular values of k as follows.

$D^1S(G)$= degree sum matrix, $DS(G)$ [19],
$D^2S(G)$= degree square sum matrix, $DSS(G)$ [2],
$D^3S(G)$= degree cube sum matrix.
$\vdots \qquad \vdots \qquad \vdots$
$D^kS(G)$= degree k^{th} power sum matrix.

Let I be the identity matrix and J be the matrix whose all entries are equal to 1. The *degree k^{th} power sum polynomial* of a graph G is defined as

$$P_{D^kS(G)}(\xi) = det(\xi I - D^kS(G)).$$

The *eigenvalues* of $D^kS(G)$, are denoted by $\xi_1, \xi_2, ..., \xi_n$, are called the *degree k^{th} power sum eigenvalues* of G and their collection is called the *degree k^{th} power sum spectra* of G. The sum of the absolute values of the degree k^{th} power sum eigenvalues of G is called the *degree k^{th} power sum energy* of G and is denoted by $E_{D^kS}(G)$. Thus

$$E_{D^kS}(G) = \sum_{i=1}^{n} |\xi_i|. \tag{1.1}$$

Example 1.1. Let K_3 be a complete graph (see Figure 1).

$$D^2S(K_3) = \begin{pmatrix} 0 & 8 & 8 \\ 8 & 0 & 8 \\ 8 & 8 & 0 \end{pmatrix}$$

$P_{D^2S(K_3)}(\xi) = \xi^3 - 192\xi - 1024$.
The eigen values of $D^2S(K_3)$ are 16, -8 and -8. Therefore, $E_{D^2S}(K_3) = 32$.

$$D^3S(K_3) = \begin{pmatrix} 0 & 16 & 16 \\ 16 & 0 & 16 \\ 16 & 16 & 0 \end{pmatrix}$$

$P_{D^3S(K_3)}(\xi) = \xi^3 - 768\xi - 8192$.
The eigen values of $D^3S(K_3)$ are 32, -16 and -16. Therefore, $E_{D^3S}(K_3) = 64$.

2. Degree k^{th} power sum polynomial of some graphs

In this section, we obtain degree k^{th} power sum polynomial of some graphs. We require following lemma in order to prove the forthcoming theorems.

Lemma 2.1. [20] *If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\xi + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\xi + b)I_{n_2} - bJ_{n_2} \end{vmatrix} \tag{2.1}$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\xi + a)^{n_1-1}(\xi + b)^{n_2-1}\{[\xi - (n_1 - 1)a][\xi - (n_2 - 1)b] - n_1n_2cd\}.$$

Theorem 2.2. *The degree k^{th} power sum polynomial of the complete bipartite graph K_{n_1,n_2} is*

$$P_{D^kS(K_{n_1,n_2})}(\xi) = (\xi + 2n_2^k)^{n_1-1}(\xi + 2n_1^k)^{n_2-1}\Big\{\big[\xi^2 + 4n_1^kn_2^k(n_1 - 1)(n_2 - 1) - 2\xi\big(n_2^k(n_1 - 1) + n_1^k(n_2 - 1)\big)\big]\Big\}.$$

Proof. The degree k^{th} power sum matrix of K_{n_1,n_2} is

$$D^kS(K_{n_1,n_2}) = \begin{bmatrix} 2n_2^k(J_{n_1} - I_{n_1}) & (n_1^k + n_2^k)J_{n_1\times n_2} \\ (n_1^k + n_2^k)J_{n_2\times n_1} & 2n_1^k(J_{n_2} - I_{n_2}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{D^kS(K_{n_1,n_2})}(\xi) &= |\xi I - D^kS(K_{n_1,n_2})| \\ &= \begin{vmatrix} (\xi + 2n_2^k)I_{n_1} - (2n_2^k)J_{n_1} & -((n_1^k + n_2^k)J_{n_1\times n_2} \\ -((n_1^k + n_2^k)J_{n_2\times n_2} & (\xi + 2n_1^k)I_{n_2} - (2n_1^k)J_{n_2}) \end{vmatrix} \end{aligned}$$

Using Lemma 2.1, we get the required result. □

Corollary 2.3. *Let $S_n = K_{1,n-1}$ be the star graph with n vertices. Then*

$$P_{D^kS(S_n)}(\xi) = \big(\xi + 2\big)^{n-2}\Big(\xi^2 - 2(n-2)\xi - (n-1)\Big(1 + (n-1)^k\Big)^2\Big).$$

Theorem 2.4. *Let P_n be a path with $n \geq 3$ vertices. Then the degree k^{th} power sum polynomial of P_n is*

$$P_{D^kS(P_n)}(\xi) = (\xi{+}2^{k+1})^{n-3}(\xi{+}2)\Big(\xi^2{-}2^{k+1}(n{-}3)\xi{-}2\xi{+}2^{k+1}(n{-}3){-}2(n{-}2)(1{+}2^k)^2\Big).$$

Proof. The characteristic polynomial of degree k^{th} power sum matrix of P_n is

$$P_{D^kS(P_n)}(\xi) \quad = \quad |\xi I - D^kS(P_n)|$$

$$= \begin{vmatrix} \xi & -(1^k+2^k) & -(1^k+2^k) & \cdots & -(1^k+2^k) & -(1^k+1^k) \\ -(1^k+2^k) & \xi & -(2^k+2^k) & \cdots & -(2^k+2^k) & -(1^k+2^k) \\ -(1^k+2^k) & -(2^k+2^k) & \xi & \cdots & -(2^k+2^k) & -(1^k+2^k) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(1^k+2^k) & -(2^k+2^k) & -(2^k+2^k) & \cdots & \xi & -(1^k+2^k) \\ -(1^k+1^k) & -(1^k+2^k) & -(1^k+2^k) & \cdots & -(1^k+2^k) & \xi \end{vmatrix} \tag{2.2}$$

Substract second column from the columns 3, 4,...,n-1 and substract first column from the last column of Eq. 2.2 to get Eq. 2.3.

$$= \begin{vmatrix} \xi & -(1+2^k) & 0 & \cdots & 0 & -2-\xi \\ -(1+2^k) & \xi & -2^{k+1}-\xi & \cdots & -2^{k+1}-\xi & 0 \\ -(1+2^k) & -2^{k+1} & \xi+2^{k+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(1+2^k) & -2^{k+1} & 0 & \cdots & \xi+2^{k+1} & 0 \\ -2 & -(1+2^k) & 0 & \cdots & 0 & \xi+2 \end{vmatrix} \tag{2.3}$$

Adding last row to the first row and then adding rows 3, 4,...,n-1 to the second row of Eq. 2.3, we get Eq. 2.4.

$$= \begin{vmatrix} \xi-2 & -2(1+2^k) & 0 & \cdots & 0 & 0 \\ -(1+2^k)(n-2) & \xi-2^{k+1}(n-3) & 0 & \cdots & 0 & 0 \\ -(1+2^k) & -2^{k+1} & \xi+2^{k+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(1+2^k) & -2^{k+1} & 0 & \cdots & \xi+2^{k+1} & 0 \\ -2 & -(1+2^k) & 0 & \cdots & 0 & \xi+2 \end{vmatrix} \tag{2.4}$$

Eq. 2.4 can be reduced to

$$P_{D^kS(P_n)}(\xi) \quad = \quad (\xi+2^{k+1})^{n-3}(\xi+2)\begin{vmatrix} \xi-2 & -2(1+2^k) \\ -(1+2^k)(n-2) & \xi-2^{k+1}(n-3) \end{vmatrix}$$

$$P_{D^kS(P_n)}(\xi) = (\xi+2^{k+1})^{n-3}(\xi+2)\Big(\xi^2-2^{k+1}(n-3)\xi-2\xi+2^{k+2}(n-3)-2(n-2)(1+2^k)^2\Big).$$

□

Definition 2.5. [13] For $n \geq 4$, the wheel W_n is defined to be the graph K_1+C_{n-1}.

Theorem 2.6. *The degree k^{th} power sum polynomial of the wheel graph W_n is*

$$P_{D^kS(W_n)}(\xi) = (\xi+2\cdot 3^k)^{n-2}\{\xi[\xi-(n-2)2\cdot 3^k]-(n-1)(n^k+3^k)^2\}.$$

Proof. The degree k^{th} power sum matrix of W_n is

$$D^kS(W_n) = \begin{bmatrix} 0(J_1-I_1) & (n^k+3^k)J_{1\times n-1} \\ (n^k+3^k)J_{n-1\times 1} & 2\cdot 3^k(J_{n-1}-I_{n-1}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{D^kS(W_n)}(\xi) &= |\xi I - D^kS(W_n)| \\ &= \begin{vmatrix} (\xi+0)I_1 - 0J_1 & -(n^k+3^k)J_{1\times n-1} \\ -(n^k+3^k)J_{n-1\times 1} & (\xi+2\cdot 3^k)I_{n-1} - (2\cdot 3^k)J_{n-1} \end{vmatrix} \end{aligned}$$

Using Lemma 2.1, we get the required result. □

Theorem 2.7. *If G is an r-regular graph with n vertices, then the degree k^{th} power sum polynomial is,*

$$P_{D^kS(G)}(\xi) = \big(\xi - 2r^k(n-1)\big)\big(\xi + 2r^k\big)^{n-1}. \tag{2.5}$$

Proof. Let G be a simple connected r-regular graph with n vertices $v_1, v_2, ..., v_n$ and let $d_i = r^k$ be the degree of v_i, $i = 1, 2, ..., n$. Then,

$$d^k s_{ij} = \begin{cases} d_i^k + d_j^k = 2r^k, & if\ i \neq j \\ 0, & otherwise. \end{cases}$$

The degree k^{th} power sum polynomial of G is

$$\begin{aligned} P_{D^kS(G)}(\xi) &= |\xi I - D^kS(G)| \\ &= |\xi I - 2r^k(A(K_n))| \\ &= (2r^k)^n \left| \frac{\xi}{2r^k} I - A(K_n) \right| \\ &= (2r^k)^n \left[\frac{\xi}{2r^k} - n + 1 \right] \left[\frac{\xi}{2r^k} + 1 \right]^{n-1} \\ &= \big(\xi - 2r^k(n-1)\big)\big(\xi + 2r^k\big)^{n-1}. \end{aligned}$$

□

Corollary 2.8. *For a complete graph K_n, the degree k^{th} power sum polynomial is,*

$$P_{D^kS(K_n)}(\xi) = \big(\xi + 2(n-1)^k\big)^{n-1}\big(\xi - 2(n-1)^{k+1}\big).$$

Proof. The complete graph K_n is $(n-1)$-regular graph with n vertices. Therefore, the result follows from Eq. 2.5. □

Corollary 2.9. *If G is a cycle C_n, then the degree k^{th} power sum polynomial is,*

$$P_{D^kS(C_n)}(\xi) = \big(\xi + 2^{k+1}\big)^{n-1}\big(\xi - 2^{k+1}(n-1)\big).$$

Proof. The cycle C_n is 2-regular graph with n vertices. Therefore, the result follows from Eq. 2.5. □

Definition 2.10. [13] The *complement* of a graph G is a graph $\overline{G}$ with vertex set $V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if these are not adjacent in G.

If G is an r-regular graph with n vertices then $\overline{G}$ is an $(n-r-1)$-regular graph on the same number of vertices.

Theorem 2.11. *If G is an r-regular graph with n vertices, then the degree k^{th} power sum polynomial of its complement of a graph $\overline{G}$ is*

$$P_{D^kS(\overline{G})}(\xi) = \big(\xi + 2(n-r-1)^k\big)^{n-1}\big(\xi - 2(n-r-1)^k(n-1)\big).$$

Proof. If G is r-regular, then $\overline{G}$ is $(n-r-1)$-regular graph. Therefore by Eq. 2.5, we get the desired result. □

Definition 2.12. [13] The k^{th} iterated line graph $L^k(G)$ of G is defined as $L^k(G) = L(L^k(G))$, $k = 1, 2, ...$, where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$.

Corollary 2.13. *Let G is an r-regular graph of order n and n_k be the order of $L^k(G)$. The degree k^{th} power sum polynomial of k^{th} iterated line graph $L^k(G)$ is* $P_{D^kS(L^k(G))}(\xi) = \big(\xi + 2(2^kr - 2^{k+1} + 2)^k\big)^{n-1}\big(\xi - 2(2^kr - 2^{k+1} + 2)^k(n-1)\big)$.

Proof. The k^{th} iterated line graph $L^k(G)$ is a regular graph of degree $2^kr-2^{k+1}+2$. Therefore by Eq. 2.5, we get the desired result. □

Definition 2.14. [8] The jump graph $J(G)$ of a graph G is a graph with vertex set as $E(G)$ where the two vertices of $J(G)$ are adjacent if and only if they correspond to two nonadjacent edges of G.

Corollary 2.15. *If G is an r-regular graph of order n, then*
$$P_{D^kS(J(G))}(\xi) = \left(\xi + 2\left(\frac{(n-4)r}{2} + 1\right)^k\right)^{\frac{nr}{2}-1}\left(\xi - 2\left(\frac{(n-4)r}{2} + 1\right)^k\left(\frac{nr}{2} - 1\right)\right).$$

Proof. The jump graph of an r-regular graph is $\left(\frac{(n-4)r}{2} + 1\right)$-regular graph with $\frac{nr}{2}$ vertices. Therefore, the result follows from Eq. 2.5.

□

Definition 2.16. [13] The n-cube Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. The Q_n has 2^n vertices which may be labelled $a_1, a_2, ..., a_n$, where each a_i is either 0 or 1. Two points of Q_n are adjacent if their binary representations differ at exactly one place.

Corollary 2.17. *If n-cube Q_n is n-regular with 2^n vertices and $n2^{n-1}$ edges, then*

$$P_{D^kS(Q_n)}(\xi) = \big(\xi + 2n^k\big)^{2^n-1}\big(\xi - 2n^k(2^n-1)\big).$$

Proof. The n-cube graph of an r-regular graph is n-regular graph with 2^n vertices. Therefore, the result follows from Eq. 2.5. □

3. Degree k^{th} power sum polynomial of generalized graphs related to cycle

In this section, we obtain the degree k^{th} power sum polynomial of generalized graphs related to cycle. Scaria et al., [21] defined some generalised graphs related to the cycle C_n namely, $M(n,k)$, $T(n,k,t)$, $N(n,k,t)$, $D(n,k,t,m)$, $C(n,k,t)$, $PC(n,k,t)$ and $CC(n,k,t)$.

Definition 3.1. [21] The graph $M(n,k)$ has vertices, respectively, edges given by

$$V(M(n,k)) = \{u_i, v_i/0 \leq i \leq n-1\},$$

$$E(M(n,k)) = \{u_iv_i, u_iv_{i+1}, v_iv_{i+k}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $k < \frac{n}{2}$.

Definition 3.2. [21] The graph $T(n,k,t)$ has vertices, respectively, edges given by

$$V(T(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$

$$E(T(n,k,t)) = \{u_iv_i, u_iv_{i+1}, u_iu_{i+t}, v_iv_{i+k}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Definition 3.3. [21] The graph $N(n,k,t)$ has vertices, respectively, edges given by

$$V(N(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$

$$E(N(n,k,t)) = \{u_iv_{i+t}, v_iv_{i+k}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Scaria et al., [21] gave the definition of $N(n,k,t)$ and mentioned that $N(n,k,t)$ is the neighborhood corona of C_n with K_1, for $k = t = 1$. There is some flaw in the definition, we correct it.

Definition 3.4. The graph $N(n,k,t)$ has vertices, respectively, edges given by

$$V(N(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$

$$E(N(n,k,t)) = \{u_iv_{i+t}, v_iu_{i+t}, v_iv_{i+k}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Definition 3.5. [21] The graph $D(n,k,t,m)$ has vertices, respectively, edges given by

$$V(D(n,k,t,m)) = \{u_i, v_i/0 \leq i \leq n-1\},$$

$$E(D(n,k,t,m)) = \{u_iu_{i+t}, u_iv_{i+m}, v_iv_{i+k}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $t, m, k < \frac{n}{2}$.

Scaria et al., [21] gave the definition of $D(n,k,t,m)$ and mentioned that $D(n,k,t,m)$ is the double graph of C_n, for $k = t = m = 1$. There is some flaw in the definition, we give correct definition.

Definition 3.6. The graph $D(n,k,t,m)$ has vertices, respectively, edges given by

$$V(D(n,k,t,m)) = \{u_i, v_i/0 \leq i \leq n-1\},$$

$$E(D(n,k,t,m)) = \{u_iu_{i+t}, u_{i+m}v_i, u_iv_{i+m}, v_iv_{i+k}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $t, m, k < \frac{n}{2}$.

Definition 3.7. [21] The graph $C(n,k,t)$ has vertices, respectively, edges given by

$$V(C(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$
$$E(C(n,k,t)) = \{u_iu_{i+k}, u_iv_i, v_iv_{i+t}/0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Definition 3.8. [21] The graph $PC(n,k,t)$ has vertices, respectively, edges given by

$$V(PC(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$
$$E(PC(n,k,t)) = \{u_iv_j, u_iv_i, v_iv_{i+k}/0 \leq i, j \leq n-1\},$$

where $j \notin \{i+n-t, i+t\}$ and the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Scaria et al., [21] gave the definition of $PC(n,k,t)$. There is a flaw in the definition, we give correct definition.

Definition 3.9. [21] The graph $PC(n,k,t)$ has vertices, respectively, edges given by

$$V(PC(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$
$$E(PC(n,k,t)) = \{u_iv_j, v_iv_{i+k}/0 \leq i, j \leq n-1\},$$

where $j \notin \{i+n-t, i+t\}$ and the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Definition 3.10. [21] The graph $CC(n,k,t)$ has vertices, respectively, edges given by

$$V(CC(n,k,t)) = \{u_i, v_i/0 \leq i \leq n-1\},$$
$$E(CC(n,k,t)) = \{u_iu_j, u_iv_i, v_iv_{i+k}/0 \leq i, j \leq n-1\},$$

where $j \notin \{i+n-t, i+t\}$ and the subscripts are expressed as integers modulo $n > 4$ and $t, k < \frac{n}{2}$.

Theorem 3.11. *The degree k^{th} power sum polynomial of $M(n,k)$ is*

$$\begin{aligned} P_{D^kS(M(n,k))}(\xi) &= (\xi + 2^{2k+1})^{n-1}(\xi + 2^{k+1})^{n-1}\{\xi^2 + (n-1)^2 2^{3k+2} \\ &\quad -(n-1)(2^{2k+1} + 2^{k+1}) - n^2(4^k + 2^k)^2\}. \end{aligned}$$

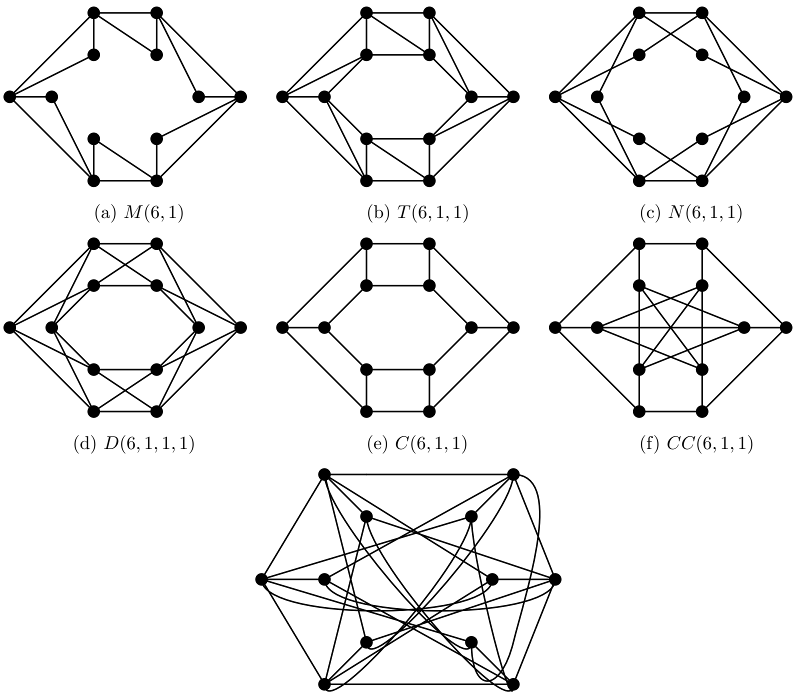
Proof. The degree k^{th} power sum matrix of $M(n,k)$ is

$$D^kS(M(n,k)) = \begin{bmatrix} 2^{2k+1}(J_n - I_n) & (4^k + 2^k)J_{n\times n} \\ (4^k + 2^k)J_{n\times n} & 2^{k+1}(J_n - I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{D^kS(M(n,k))}(\xi) &= |\xi I - D^kS(M(n,k))| \\ &= \begin{vmatrix} (\xi + 2^{2k+1})I_n - 2^{2k+1}J_n & -(4^k + 2^k)J_{n\times n} \\ -(4^k + 2^k)J_{n\times n} & (\xi + 2^{k+1})I_n - (2^{k+1})J_n \end{vmatrix} \end{aligned}$$

Using Lemma 2.1, we get the required result. □



(a) $M(6,1)$ (b) $T(6,1,1)$ (c) $N(6,1,1)$

(d) $D(6,1,1,1)$ (e) $C(6,1,1)$ (f) $CC(6,1,1)$

(g) $PC(6,1,1)$

FIGURE 2. Some generalized graphs related to cycle.

Theorem 3.12. *The degree k^{th} power sum polynomial of $T(n,k,t)$ is*

$$P_{D^kS(T(n,k,t))}(\xi) = \big(\xi + 2(4)^k\big)^{2n-1}\big(\xi - 2(4)^k(2n-1)\big).$$

Proof. The graph $T(n,k,t)$ is a 4-regular graph with $2n$ vertices. Therefore, the result follows from Eq. 2.5. □

Theorem 3.13. *The degree k^{th} power sum polynomial of $C(n,k,t)$ is*

$$P_{D^kS(C(n,k,t))}(\xi) = \big(\xi + 2(3)^k\big)^{2n-1}\big(\xi - 2(3)^k(2n-1)\big).$$

Proof. The graph $C(n,k,t)$ is a 3-regular graph with $2n$ vertices. Therefore, the result follows from Eq. 2.5. □

Theorem 3.14. *The degree k^{th} power sum polynomial of $PC(n,k,t)$ is*

$$\begin{aligned} P_{D^kS(PC(n,k,t))}(\xi) \;\; &= \;\; (\xi + 2n^k)^{n-1}(\xi + 2(n-2)^k)^{n-1}\{\xi^2 + 4(n-1)^2 n^k (n-2)^k \\ &\qquad -2(n-1)(n^k + (n-2)^k) - n^2(n^k + (n-2)^k)^2\}. \end{aligned}$$

Proof. The degree k^{th} power sum matrix of $PC(n,k,t)$ is

$$D^kS(PC(n,k,t)) = \begin{bmatrix} 2n^k(J_n - I_n) & (n^k + (n-2)^k)J_{n\times n} \\ (n^k + (n-2)^k)J_{n\times n} & 2(n-2)^k(J_n - I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{D^kS(PC(n,k,t))}(\xi) &= |\xi I - D^kS(PC(n,k,t))| \\ &= \begin{vmatrix} (\xi+2n^k)I_n - 2n^kJ_n & -(n^k+(n-2)^k)J_{n\times n} \\ -(n^k+(n-2)^k)J_{n\times n} & (\xi+2(n-2)^k)I_n-(2(n-2)^k)J_n \end{vmatrix} \end{aligned}$$

Using Lemma 2.1, we get the required result. ☐

Theorem 3.15. *The degree k^{th} power sum polynomial of $CC(n,k,t)$ is*

$$\begin{aligned} P_{D^kS(CC(n,k,t))}(\xi) &= (\xi+2\cdot 3^k)^{n-1}(\xi+2(n-2)^k)^{n-1}\{\xi^2+4\cdot 3^k(n-1)^2(n-2)^k \\ &\quad -2(n-1)(3^k-(n-2)^k)\}. \end{aligned}$$

Proof. The degree k^{th} power sum matrix of $CC(n,k,t)$ is

$$D^kS(CC(n,k,t)) = \begin{bmatrix} 2\cdot 3^k(J_n-I_n) & (3^k+(n-2)^k)J_{n\times n} \\ (3^k+(n-2)^k)J_{n\times n} & 2(n-2)^k(J_n-I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{D^kS(CC(n,k,t))}(\xi) &= |\xi I - D^kS(CC(n,k,t))| \\ &= \begin{vmatrix} (\xi+2\cdot 3^k)I_n - 2\cdot 3^kJ_n & -(3^k+(n-2)^k)J_{n\times n} \\ -(3^k+(n-2)^k)J_{n\times n} & (\xi+2(n-2)^k)I_n-(2(n-2)^k)J_n \end{vmatrix} \end{aligned}$$

Using Lemma 2.1, we get the required result. ☐

Theorem 3.16. *The degree k^{th} power sum polynomial of $N(n,k,t)$ is*

$$\begin{aligned} P_{D^kS(N(n,k,t))}(\xi) &= (\xi+2^{2k+1})^{n-1}(\xi+2^{k+1})^{n-1}\{\xi^2+2^{3k+2}(n-1)^2 \\ &\quad -(n-1)(2^{2k+1}+2^{k+1})-n^2(4^k+2^k)^2\}. \end{aligned}$$

Proof. The degree k^{th} power sum matrix of $N(n,k,t)$ is

$$D^kS(N(n,k,t)) = \begin{bmatrix} 2^{2k+1}(J_n-I_n) & (4^k+2^k)J_{n\times n} \\ (4^k+2^k)J_{n\times n} & 2^{k+1}(J_n-I_n) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{D^kS(N(n,k,t))}(\xi) &= |\xi I - D^kS(N(n,k,t))| \\ &= \begin{vmatrix} (\xi+2^{2k+1})I_n - 2\cdot 3^kJ_n & -(4^k+2^k)J_{n\times n} \\ -(4^k+2^k)J_{n\times n} & (\xi+2^{k+1})I_n-(2^{k+1})J_n \end{vmatrix} \end{aligned}$$

Using Lemma 2.1, we get the required result. ☐

Theorem 3.17. *The degree k^{th} power sum polynomial of $D(n,k,t,m)$ is*

$$P_{D^kS(D(n,k,t,m))}(\xi) = \left(\xi+2(4)^k\right)^{2n-1}\left(\xi-2(4)^k(2n-1)\right).$$

Proof. The graph $D(n,k,t,m)$ is a 4-regular graph with $2n$ vertices. Therefore, the result follows from Eq. 2.5. ☐

4. Bounds for the largest degree k^{th} power sum eigenvalue and degree k^{th} power sum energy of graphs

We require the following theorems to prove our next results.

Theorem 4.1. [4] *The Cauchy-Schwarz inequality states that if* $(a_1, a_2, ..., a_n)$ *and* $(b_1, b_2, ..., b_n)$ *are two real n-vectors, then*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big).$$

Theorem 4.2. [17] *Let* a_i *and* b_i, $1 \leq i \leq n$ *are nonnegative real numbers, then*

$$\sum_{i=1}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2 - \Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2,$$

where $M_1 = \max\limits_{1\leq i\leq n}(a_i)$; $M_2 = \max\limits_{1\leq i\leq n}(b_i)$; $m_1 = \min\limits_{1\leq i\leq n}(a_i)$; $m_2 = \min\limits_{1\leq i\leq n}(b_i)$.

Theorem 4.3. [5] *Let* a_i *and* b_i, $1 \leq i \leq n$ *are nonnegative real numbers, then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} b_i\Big)\right| \leq \mu(n)(A-a)(B-b),$$

where a, b, A *and* B *are real constants, such that for each* i, $1 \leq i \leq n$, $a \leq a_i \leq A$ *and* $b \leq b_i \leq B$. *Further,* $\mu(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

Theorem 4.4. [11] *Let* a_i *and* b_i, $1 \leq i \leq n$ *are nonnegative real numbers, then*

$$\sum_{i=1}^{n} {b_i}^2 + c_1c_2 \sum_{i=1}^{n} {a_i}^2 \leq (c_1 + c_2)\left(\sum_{i=1}^{n} a_i b_i\right).$$

where c_1 *and* c_2 *are real constants, such that for each* i, $1 \leq i \leq n$ *holds,* $c_1 a_i \leq b_i \leq c_2 a_i$.

Theorem 4.5. *The eigen values of* $D^kS(G)$ *satisfies the relations*

1. $\sum_{i=1}^{n} \xi_i = 0$.
2. $\sum_{i=1}^{n} \xi_i^2 = 2S$, *where* $S = \sum_{i<j}(d_i^k + d_j^k)^2$.

Proof. By the definition of $D^kS(G)$, $\sum_{i=1}^{n}\xi_i = 0$
Next,

$$\begin{aligned}\sum_{i=1}^{n}\xi_i^2 &= trace((D^kS(G))^2)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}c_{ji}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}^2\\ &= 2\sum_{i<j}(d_i^k+d_j^k)^2\\ &= 2S,\ where\ S=\sum_{i<j}(d_i^k+d_j^k)^2.\end{aligned}$$

□

Theorem 4.6. *If G is a connected graph with n vertices, then*

$$\xi_1 \le \sqrt{\frac{2S(n-1)}{n}}.$$

Proof. Let $a_i = 1$ and $b_i = \xi_i$ for $i = 2, 3, ..., n$ in Theorem 4.1. Therefore

$$\Big(\sum_{i=2}^{n}\xi_i\Big)^2 \le (n-1)\Big(\sum_{i=2}^{n}{\xi_i}^2\Big) \tag{4.1}$$

From Theorem 4.5, we have $\sum_{i=2}^{n}\xi_i = -\xi_1$ and $\sum_{i=2}^{n}{\xi_i}^2 = 2S - {\xi_1}^2$
Therefore Eq. 4.1 becomes

$$(-\xi_1)^2 \le (n-1)(2S-{\xi_1}^2)$$

which gives

$$\xi_1 \le \sqrt{\frac{2S(n-1)}{n}}.$$

□

Equality holds for ξ_1 if G is regular graph.

Theorem 4.7. *If G is a graph with n vertices, then*

$$\sqrt{2S} \le E_{D^kS}(G) \le \sqrt{2nS}.$$

Proof. For $a_i = 1$ and $b_i = \xi_i$ in Theorem 4.1 we obtain,

$$\Big(\sum_{i=1}^{n}\xi_i\Big)^2 \le n\Big(\sum_{i=1}^{n}|\xi_i|^2\Big)$$

Now

$$(E_{D^kS}(G))^2 \le 2nS$$
$$E_{D^kS}(G) \le \sqrt{2nS} \tag{4.2}$$

This is an upper bound. Now

$$(E_{D^kS}(G))^2 = \Big(\sum_{i=1}^{n} \xi_i\Big)^2 \geq \sum_{i=1}^{n} |\xi_i|^2 = 2S,$$

which gives a lower bound

$$E_{D^kS}(G) \geq \sqrt{2S}. \tag{4.3}$$

Combining Eq. 4.2 and Eq. 4.3 we get

$$\sqrt{2S} \leq E_{D^kS}(G) \leq \sqrt{2nS}.$$

□

Theorem 4.8. *If G is a graph with n vertices, then*

$$E_{D^kS}(G) \geq \sqrt{2nS - \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2}.$$

Proof. For $a_i = 1$ and $b_i = \xi_i$ in Theorem 4.2 we get,

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\xi_i|^2 - (\sum_{i=1}^{n} |\xi_i|)^2 \leq \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2$$

$$2nS - (E_{D^kS}(G))^2 \leq \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2$$

$$E_{D^kS}(G) \geq \sqrt{2nS - \frac{n^2}{4}(|\xi_1| - |\xi_n|)^2}.$$

□

Theorem 4.9. *If G is an (n,m) graph, then*

$$E_{D^kS}(G) \geq \sqrt{2nS - \mu(n)(|\xi_1| - |\xi_n|)^2},$$

where $\mu(n) = n\lfloor \frac{n}{2} \rfloor \Big(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\Big)$.

Proof. Let $a_i = |\xi_i| = b_i$, $A = \xi_1 = B$ and $a = |\xi_n| = b$ in Theorem 4.3 then

$$\left| n\sum_{i=1}^{n} |\xi_i|^2 - \Big(\sum_{i=1}^{n} |\xi_i|\Big)^2 \right| \leq \mu(n)\Big(|\xi_1| - |\xi_n|\Big)^2 \tag{4.4}$$

Since, $E_{D^kS}(G) = \sum_{i=1}^{n} |\xi_i|$ and $\Big(\sum_{i=1}^{n} |\xi_i|\Big)^2$
Inequality 4.4 gives

$$2nS - (E_{D^kS}(G))^2 \leq \mu(n)(|\xi_1| - |\xi_n|)^2$$

.

$$E_{D^kS}(G) \geq \sqrt{2nS - \mu(n)(|\xi_1| - |\xi_n|)^2}.$$

□

Theorem 4.10. *If G is a graph with n vertices, then*

$$E_{D^kS}(G) \geq \frac{2S + n|\xi_1||\xi_n|}{|\xi_1| + |\xi_n|},$$

where $|\xi_1|$ is maximum and $|\xi_n|$ is minimum absolute value of $\xi_i^{\prime}s$.

Proof. Let $a_i = 1$ and $b_i = \xi_i$, $c_1 = |\xi_n|$ and $c_2 = |\xi_1|$ in Theorem 4.4 then

$$\sum_{i=1}^{n} |\xi_i|^2 + |\xi_1||\xi_n| \sum_{i=1}^{n} 1^2 \leq (|\xi_1| + |\xi_n|)\Big(\sum_{i=1}^{n} |\xi_i|\Big)$$

Since $E_{D^kS}(G) = \sum_{i=1}^{n} |\xi_i|$ and $\sum_{i=1}^{n} |\xi_i|^2 = 2S$.
On simplification,

$$2S + |\xi_1||\xi_n| \leq (|\xi_1| + |\xi_n|)E_{D^kS}(G) \tag{4.5}$$

simple calculation of inequality 4.5 yields the required result. □

5. Conclusion

In this paper, we have introduced a new graph matrix called the degree k^{th} power sum matrix and its energy. Further, we obtained degree k^{th} power sum polynomial of some standard class of graphs and generalized graphs related to cycle. Being new generalized version of degree based graph energy, the $D^kS(G)$ lies on the fact that their special cases, for particular chosen value of the parameter k, coincide with the some of previously considered matrices defined on graph. The bounds for the largest degree k^{th} power sum eigenvalue and degree k^{th} power sum energy of a graph are investigated.

References

1. M. Aouchiche, P. Hansen, *Distance spectra of graphs: A survey*, Linear Algebra Appl. **458** (2014), 301–386.
2. B. Basavanagoud, E. Chitra, *Degree square sum energy of graphs*, Int. J. Math. Appl. **6(2-B)** (2018), 193–205.
3. B. Basavanagoud, P. Jakkannavar, *Minimum degree energy of graphs*, Electronic Journal of Mathematical Analysis and Applications, **7(2)** (2019), 230–243.
4. S. Bernard, J. M. Child, Higher Algebra, Macmillan India Ltd., New Delhi (2001).
5. M. Biernacki, H. Pidek and C. Ryll-Nardzewsk, Sur une iné galité entre des intégrales définies, Univ. Marie Curie-Sklodowska, **A(4)** (1950), 1–4.
6. H. S. Boregouda, R. B. Jummannaver, *Neighbors degree sum energy of graphs*, Journal of Applied Mathematics and Computing, **67** (2021), 579–603.
7. A. E. Brouwer, W. H. Haemers, *Spectra of Graphs*, Springer, Berlin, (2012).
8. G. Chartrand, H. Hevia, E. B. Jarrett, M. Schultz, *Subgraph distances in graphs defined by edge transfers*, Discrete Math. **170** (1997), 63–79.
9. D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs-Theory and Applications*, Academic Press, New York, (1980).
10. D. Cvetković, P. Rowlinson, S. K. Simić, *Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math. (Beograd) **81** (2007), 11–27.
11. J. B. Diaz, F. T. Metcalf, *Stronger forms of a class of inequalities of G. Pólya-G. Szegö and L. V. Kantorovich*, Bull. Amer. Math. Soc. **69(3)** (1963), 415–418.
12. I. Gutman, *The energy of a graph*, Ber. Math-Statist. Sekt. Forschungszentrum Graz, **103** (1978), 1–22.
13. F. Harary, *Graph Theory*, Addison-Wesely, Reading, Mass (1969).
14. G. Indulal, A. Vijayakumar, *A note on energy of some graphs*, MATCH Commun. Math. Comput. Chem. **59** (2008), pp. 269–274.
15. S. R. Jog, S. P. Hande, D. S. Revankar, *Degree sum polynomial of graph valued functions on regular graphs*, Int. J. Graph Theory, **1** (2013), 108–115.

16. B. Mohar, *The Laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. R. Ollermann and A. J. Schwenk (Eds.), Graph Theory, Combinatorics and Applications, Wiley, New York, (1991), pp.871–898.
17. N. Ozeki, *On the estimation of inequalities by maximum and minimum values*, J. College Arts and Sci., Chiba Univ., **5** (1968), 199–203.
18. H. S. Ramane, S. B. Gudimani, S. S. Shinde, *Signless Laplacian polynomial and characteristic polynomial of a graph*, J. Discrete Math. (2013), Article ID 105624, 4 pages. http://dx.doi.org/10.1155/2013/105624.
19. H. S. Ramane, D. S. Revankar, J. B. Patil, *Bounds for the degree sum eigenvalues and degree sum energy of a graph*, Int. J. Pure Appl. Math. Sci. **6** (2013), 161–167.
20. H. S. Ramane, S. S. Shinde, *Degree exponent polynomial of graphs obtained by some graph operations*, Electronic Notes in Discrete Mathematics, **63** (2017), 161–168.
21. D. C. Scaria, G. Indulal, *The spectrum of some generalized graphs related to cycle*, Indian J. Pure Appl. Math. **52** (2021), 274–280.

[1]Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India
Email address: b.basavanagoud@gmail.com

[2]Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India
Email address: shrutipatil300@gmail.com

[3] #134/8, Achuta Nivas, More Galli, Cowl Bazar, Ballari - 583 102,, Karnataka, India
Email address: bkvpooja@gmail.com