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A CHARACTERIZATION OF 2-SUMMING OPERATORS IN KREIN SPACES

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ABSTRACT. Let A be a bounded linear operator and absolutely 2-summing on a Krein space $\mathcal{K}$, with canonical decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{pmatrix}.$$

In this paper, we prove that the elements in the matrix decomposition of A are absolutely 2-summing with $\pi_2(A_{ij}) \leq \dfrac{\Gamma + \gamma}{(2Re(\Gamma\overline{\gamma}))^{\frac{1}{2}}} \parallel A_{ij} \parallel \pi_2(A)$ for each $Aij \neq 0$. We characterize 2-summing operators in Krein spaces using a fixed fundamental decomposition.

1. Introduction and preliminaries

1.1. **Summing Operators.** The theory of absolutely summing operators lie in the work that was undertaken by Grothendieck [10] in 1950's. However, it was not until 1967 that Albert Pietsh clearly isolated this class of operators and establish many of their fundamental properties that we use today [14]. The appearance of the seminar paper" Absolutely summing operator in $\mathcal{L}_p$ and their applications" of Joram Lindenstrauss and Alexander Pelezyski [13] was not only a turning point in the development of absolutely summing operators but was also gave a lot of recognition to Pietsch's work. Grothendieck's inequality

$$\frac{\pi}{2} \leq k_G \leq \sinh(\frac{\pi}{2}) \tag{1.1}$$

Where k_G is Grothendiek's constant, played an important role in establishing some of the known properties of absolutely 2-summing operators that we take for granted today. This inequality first appeared in Grothendieck's paper of 1956 [10] and has attracted a great deal of attention. This has inspired several new proofs over the years. In 1981 Behnam [2] provided a proof of Schrodinger equation for self adjoint operator and all symmetric operators in Hilbert spaces with any element of the domain of the operator as initial value. For self adjoint operators, he has shown that the solution of the schrodinger equation appears as a transform of the initial value by a unitary operator. He has given necessary

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conditions for linear operators on Krein spaces with decomposition $\mathcal{K} = K_+ \oplus \mathcal{K}_-$ to be symmetric, where $\mathcal{K}_+$ is a Hilbert space and $\mathcal{K}_-$ is anti space of a Hilbert space. Let A be a closed densely defined operator on a Pontryagin space(defined in the sequel)

$$\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_- \tag{1.2}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{P}_+ \\ \mathcal{P}_- \end{pmatrix} \to \begin{pmatrix} \mathcal{P}_+ \\ \mathcal{P}_- \end{pmatrix} \tag{1.3}$$

It is shown in [2] that if the operator A_{22} is self adjoint, then for any $\phi \in D(A)$,

$$\frac{d\phi}{dt} = iA\psi \tag{1.4}$$

with $\psi(0) = \phi$ has a solution.

1.2. **Indefinite inner product spaces.**

Definition 1.1. An inner product space is a complex linear space together with a Hermitian form defined on it so that the associated quadratic form takes both positive and negative values.

Example 1.2. When one considers an orthogonal direct sum of two subspaces of a Hilbert space, -1 equipped with the original inner product and the other one with times the original inner product.

A first mathematical treatment of an indefinite inner product spaces appeared in pontryagin's paper [16] of 1944. Pontryagin who was inspired by the unpublished work of Sobolev was unaware of the investigation of Dirac around the same time [15].

Definition 1.3. Let $\mathcal{K}$ be a complex vector space. By an indefinite inner product on $\mathcal{K}$ we mean a complex valued function $\langle .,. \rangle_{\mathcal{K}}$ defined on $\mathcal{K} \times \mathcal{K}$ which satisfies the following conditions:

(i) (Linearity) $\langle \alpha x + \beta y, z \rangle_{\mathcal{K}} = \alpha \langle x, z \rangle_{\mathcal{K}} + \beta \langle y, z \rangle_{\mathcal{K}}$ for all $x, y, z \in \mathcal{K}$ and $\alpha, \beta \in \mathbb{C}$,

(ii) (Symmetry)$\langle x, y \rangle_{\mathcal{K}} = \overline{\langle y, x \rangle}_{\mathcal{K}}$ for all $x, y \in \mathcal{K}$.

The relations in (i)and (ii) imply that
$\langle \alpha x + \beta y, z \rangle_{\mathcal{K}} = \overline{\alpha}\, \overline{\langle z, x \rangle}_{\mathcal{K}} + \overline{\beta}\, \overline{\langle z, y \rangle}_{\mathcal{K}}$.

Definition 1.4. The function $\langle .,. \rangle_{\mathcal{K}}$ is called an indefinite inner product on $\mathcal{K}$. The space $\mathcal{K}$ together with an indefinite inner product, $\langle .,. \rangle_{\mathcal{K}}$ defined on it is called an indefinite inner product space. It is denoted this by $(\mathcal{K}, \langle .,. \rangle_{\mathcal{K}})$.

Definition 1.5. Let $(\mathcal{K}, \langle .,. \rangle_{\mathcal{K}})$ be an indefinite inner product space. By the anti space of $(\mathcal{K}, \langle .,. \rangle_{\mathcal{K}})$ we mean the space $(\mathcal{K}, -\langle .,. \rangle_{\mathcal{K}})$ which coincides with $\mathcal{K}$ as a vector space but with the sign of the indefinite inner product reversed.

1.3. **The reverse of the Cauchy Schwarz inequality.**

Let $\mathcal{K}$ be a Krein space and let $x, y \neq 0 \in \mathcal{K}$ and $\Gamma, \gamma \in \mathbb{C}$ be such that

$$\| x - \frac{\Gamma + \gamma}{2} y \| \leq \frac{1}{2} \mid \Gamma - \gamma \mid \| y \| \tag{1.5}$$

holds. If $Re(\Gamma\overline{\gamma}) \geq 0$ then, $\| x \| \| y \| \leq \frac{1}{4} \frac{|\Gamma + \gamma|^2}{Re(\Gamma\overline{\gamma})} \mid \langle Jx, y \rangle_{\mathcal{K}} \mid^2$.

2. Krein spaces

These are the most important type of inner product spaces and can roughly be characterized as non-degenerate, decomposable and complete inner product spaces. These are more general than Hilbert spaces which are basic examples of Krein spaces.[4]

Definition 2.1. A Krein space $\mathcal{K}$ is an indefinite inner product space $(\mathcal{K}, \langle ., . \rangle_{\mathcal{K}})$ which can be expressed as a direct orthogonal sum

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-. \tag{2.1}$$

where $\mathcal{K}_+$ is a Hilbert space and $\mathcal{K}_-$ is the anti space of a Hilbert space.

Any such representation is called a fundamental decomposition. This decomposition is not unique in general, but the components $\mathcal{K}_+$ and $\mathcal{K}_-$ are uniquely determined. If $\mathcal{K}$ is a Krein space, then the numbers ind$\pm\mathcal{K}$ = dim$\mathcal{K}\pm$ do not depend on the choice of decomposition above. We call ind$\pm\mathcal{K}$ the positive and negative indices of the space.

Definition 2.2. A Krein space $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ in which dim $\mathcal{K}_- < \infty$ is called a Pontryagin space.

Let $\mathcal{K}$ be a Krein space with two fundamental decompositions $\mathcal{K}_1 = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$ and $\mathcal{K}_2 = \mathcal{K}_2^+ \oplus \mathcal{K}_2^-$ and associated Hilbert spaces

$$\mid \mathcal{K}_1 \mid = \mathcal{K}_1^+ \oplus \mid \mathcal{K}_1^- \mid \text{ and } \mid \mathcal{K}_2 \mid = \mathcal{K}_2^+ \oplus \mid \mathcal{K}_2^- \mid \tag{2.2}$$

Then $\mid \mathcal{K}_1 \mid$ and $\mid \mathcal{K}_2 \mid$ have equivalent norms, that is , there exists $m > 0$ and $M > 0$ such that

$$m \parallel f \parallel_{|\mathcal{K}_2|} \leq \parallel f \parallel_{\mathcal{K}_1} \leq M \parallel f \parallel_{\mathcal{K}_1} \tag{2.3}$$

for all $f \in \mathcal{K}$. Hence the norm topologies of the Hilbert spaces in (2.2) are identical.

2.1. Bounded linear operators.

The notions of a bounded linear operator on any given Krein space is similar to that of a Hilbert space and in general to that of a bounded linear operator on any normed linear space.

Definition 2.3. Let X and Y be normed linear spaces and $A : D(A) \to Y$ a linear operator , where $D(A) \subset X$. The operator A is said to be bounded if there exists areal number c such that for all
$x \in D(A)$

$$\parallel Ax \parallel \leq c. \parallel x \parallel . \tag{2.4}$$

If no such a constant c exists then, we say that A is unbounded. $\Rightarrow \frac{\|Ax\|}{\|x\|} \leq c$. The smallest c is a supremum. This quantity is denoted by $\parallel A \parallel = \sup_{\substack{x \in D(A) \\ \|x\|=1}} \parallel Ax \parallel$.

Proof. We write $\parallel x \parallel = a$ and set $y = (\frac{1}{a})x$, where $x \neq 0$. Then $\parallel y \parallel = \frac{\|x\|}{a} = 1$ and since A is linear then,
$\parallel A \parallel = \sup_{\substack{x \in D(A) \\ x \neq 0}} \frac{1}{a} \parallel Ax \parallel = \sup_{\substack{x \in D(A) \\ x \neq 0}} \parallel A(\frac{1}{a})x \parallel = \sup_{\substack{y \in D(A) \\ \|y\|=1}} \parallel Ay \parallel$. Writing x for y, we get the result. $\square$

Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces. By $B(\mathcal{H}, \mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, we write $B(\mathcal{H})$ in place of $B(\mathcal{H}, \mathcal{H})$. For every $A \in B(\mathcal{H}, \mathcal{K})$, there is a unique $A^* \in B(\mathcal{K}, \mathcal{H})$ such that $\langle Af, g\rangle_{\mathcal{K}} = \langle f, A^* g\rangle_{\mathcal{H}},\ f \in \mathcal{H}, g \in \mathcal{K}$. We call A^* an adjoint operator of A. An operator $A \in B(\mathcal{H})$ is self adjoint if $A^* = A$. A linear operator A in an inner product space $\mathcal{K}$ is said to be symmetric if $\langle Ax,\, y\rangle = \langle x,\, Ay\rangle$, for every pair $x,\, y \in D(A)$.

3. Decompositions of summing operators in Krein spaces

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{M} \oplus M^{\perp}$, where $\mathcal{M}^{\perp}$ is the orthogonal compliment of $\mathcal{M}$ in $\mathcal{H}$. With this representation of $\mathcal{H}$, any bounded linear operator T on $\mathcal{H}$ can be decomposed in the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix} \tag{3.1}$$

where we write $\begin{pmatrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{pmatrix}$ in place of $\mathcal{M} \oplus \mathcal{M}^{\perp}$. Where T_{11} is an operator on $\mathcal{M}$, T_{12} is an operator from $\mathcal{M}^{\perp}$ to $\mathcal{M}$, T_{21} is an operator from $\mathcal{M}$ to $\mathcal{M}^{\perp}$, T_{22} is an operator on $\mathcal{M}^{\perp}$. Representation (3.1) of the operator T is called its decomposition with respect to the subspace $\mathcal{M}$. In general this decomposition depends on subspace $\mathcal{M}$. The situation in the Krein space is slightly different from what we have described above. A decomposition of a Krein space always exists and therefore a bounded linear operator between Krein spaces always has a decomposition.

3.1. **The symmetric property of bounded operators on Krein spaces.** If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces with fixed decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ respectively, and A is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$, then A can be uniquely represented in the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{3.2}$$

where A_{11} is an operator from $\mathcal{H}_+$ to $\mathcal{K}_+$, A_{12} is an operator from $\mathcal{H}_-$ to $\mathcal{K}_+$, A_{21} is an operator from $\mathcal{H}_+$ to $\mathcal{K}_-$ and A_{22} is an operator from $\mathcal{H}_-$ to $\mathcal{K}_-$. The representation of the operator A in (3.2) is called its canonical decomposition.

3.2. **Absolutely p-summing operators in Krien spaces.** Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces and let $(1 \leq p < \infty)$. A bounded linear operator $U : \mathcal{H} \to \mathcal{K}$ is called absolutely *p-summing* if there exists a constant $c > 0$ such that for any positive integer m and any vectors $x_1, x_2, x_3, ..., x_m$ in $\mathcal{H}$, the inequality

$$\left(\sum_{i=1}^{m} \| \, Ux_i \, \|^p \right)^{1/p} \leq c \,.\, \sup \left\{ \left(\sum_{i=1}^{m} | \, \langle x_i, y\rangle_{\mathcal{H}} \, |^p \right)^{1/p} : y \in \mathcal{H}, \| \, y \, \| \leq 1 \right\} \tag{3.3}$$

holds. The smallest c for which (4.1) holds is denoted by $\pi_p(U)$ while $\Pi_p(\mathcal{H}, \mathcal{K})$ stands for the set of all absolutely p-summing operators from $\mathcal{H}$ to $\mathcal{K}$. When $p = 1$ these operators are called *absolutely summing* operators. For the case $p = 2$, the operators are called *absolutely* 2-*summing* operators which the main focus in this paper. Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces and let h and k be fixed elements of $\mathcal{H}$ and $\mathcal{K}$ respectively with $\| h \| \leq 1$. Define an operator $U : \mathcal{H} \to \mathcal{K}$ by $U(x) = \langle x, h\rangle_{\mathcal{H}} \mathcal{K}$. Then U is clearly well defined and linear. The fact that this operator is bounded and absolutely 2-summing with $\pi_2(U) \leq \| k \|^2$, follows from

$$\| U(x) \|^2 = \| \langle x, h\rangle_{\mathcal{H}} k \|^2 = | \langle x, h\rangle_{\mathcal{H}} |^2 \| k \|^2 .$$

4. Characterization of 2-summing operators in Krein spaces

4.1. **Introduction.** Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces and let $(1 \leq p < \infty)$. A bounded linear operator $U : \mathcal{H} \to \mathcal{K}$ is called absolutely 2-*summing* if there exists a constant $c > 0$ such that for any positive integer m and any vectors $x_1, x_2, x_3, ..., x_m$ in $\mathcal{H}$, the inequality

$$\left(\sum_{i=1}^{m} \| Ux_i \|^2\right)^{1/2} \leq c \,.\, \sup\left\{\left(\sum_{i=1}^{m} | \langle x_i, y\rangle_{\mathcal{H}} |^2\right)^{1/2} : y \in \mathcal{H}, \| y \| \leq 1\right\} \quad (4.1)$$

holds.

4.2. **Characterization of** 2**-summing operators in Krein spaces.** Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces and let A: $\mathcal{H} \to \mathcal{K}$ be an absolutely 2-summing operator with matrix representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}- \end{pmatrix} . \quad (4.2)$$

Where $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ are fixed fundamental decompositions. In this section we show that operators in the decomposition to be absolutely 2-summing. We consider the case when only one of the components in the matrix representation is non zero.

4.2.1. *When* $A_{ij} \neq 0, i = j$ *and* $A_{ij} = 0, i \neq j$. we have $A_{11} \neq 0$, $A_{22} \neq 0$ and $A_{12} = A_{21} = 0$. Let $A_{11} \neq 0$. Then the operator A becomes

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} .$$

Define $x_i = x_i^+ + x_i^-$ for all $x_i \in \mathcal{H}$. Then

$$Ax_i = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_i^+ \\ x_i^- \end{pmatrix} \in \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{pmatrix} = \begin{pmatrix} A_{11}x_i^+ \\ 0 \end{pmatrix} \in \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{pmatrix} .$$

We have $Ax_i = A_{11}x_i^+ \in \mathcal{K}_+$. Now we show that A_{11} is absolutely 2-summing.

Proof. Consider $Ax_i = A_{11}x_i^+ \in \mathcal{K}_+$, we have that

$$\begin{aligned}
\left(\sum_{i=1}^{m} \| A_{11}x_i^+ \|^2\right)^{\frac{1}{2}} &= \left(\sum_{i=1}^{m} \| Ax_i \|^2\right)^{\frac{1}{2}} \\
&\leq c\,.\sup\left\{\left(\sum_{i=1}^{m} | \langle x_i, y\rangle_{\mathcal{H}} |^2\right)^{\frac{1}{2}} : y \in \mathcal{H}\,, \| y \| \leq 1\right\} \\
&\leq c\,.\sup\left\{\left(\sum_{i=1}^{m} \| x_i \|^2 \| y \|^2\right)^{\frac{1}{2}} : \| y \| \leq 1\right\} \\
&\leq c\,.\left\{\left(\sum_{i=1}^{m} \| x_i \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq c\,.\left\{\left(\sum_{i=1}^{m} \| A_{11}x_i^+ \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq c\,.\left\{\left(\sum_{i=1}^{m} \| A_{11} \|^2 \| x_i^+ \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq c\,. \| A_{11} \| \left\{\left(\sum_{i=1}^{m} \| x_i^+ \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq \| A_{11} \| .c\,.\left\{\left(\sum_{i=1}^{m} \| x_i^+ \|^2 \| y_i^+ \|^2\right)^{\frac{1}{2}} : y_i^+ = \frac{x_i^+}{\| x_i^+ \|}, y_i^+ \in \mathcal{H}_+\right\} \\
&\leq \frac{| \Gamma + \gamma |}{2(Re(\Gamma\,\overline{\gamma}))^{\frac{1}{2}}} \| A_{11} \| \pi_2(A).\sup\left\{\left(\sum_{i=1}^{m} | \langle x_i^+, y_i^+\rangle_{\mathcal{H}_+} |^2\right)^{\frac{1}{2}} : | y_i^+ | \leq 1\right\}.
\end{aligned}$$

The last inequality follows from the reverse of Cauchy Schwarz inequality. Hence A_{11} is absolutely 2-summing with $\pi_2(A_{11}) \leq \frac{|\Gamma+\gamma|}{2(Re(\Gamma\,\overline{\gamma}))^{\frac{1}{2}}} \| A_{11} \| \pi_2(A)$. □

Similarly when $A_{22} \neq 0$ and $A_{11} = A_{21} = A_{12} = 0$, A_{22} is absolutely 2-summming.

4.2.2. *When* $A_{ij} = 0, i = j$ *and* $A_{ij} \neq 0, i \neq j$. For this case, the operator A has the matrix representation $A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}$. We consider two cases:
(i) When $A_{12} \neq 0$ (ii) When $A_{21} \neq 0$ and $A_{11} = A_{22} = 0$.
When $A_{21} \neq 0$ and $A_{11} = A_{12} = A_{22}$ and we define $x_i = x_i^+ + x_i^-$ for all $x_i \in \mathcal{H}$, $Ax_i = \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{pmatrix} = \begin{pmatrix} 0 \\ A_{21}x_i^+ \end{pmatrix} \in \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}_- \end{pmatrix}$. This gives

$Ax_i = A_{21}x_i^+ \in \mathcal{K}_-$. If $\| A_{21}x_i \| \geq \| x_i \|$ for all $x_i \in \mathcal{H}$ then, the operator A_{21} is absolutely 2-summing with $\frac{|\Gamma+\gamma|}{2(Re(\Gamma\overline{\gamma}))^{\frac{1}{2}}}\pi(A_{21}) \leq \| A_{21} \| \pi_2(A)$.

Proof. Consider

$$\begin{aligned}
\left(\sum_{i=1}^{m} \| A_{21}x_i^+ \|^2\right)^{\frac{1}{2}} &= \left(\sum_{i=1}^{m} \| Ax_i \|^2\right)^{\frac{1}{2}} \\
&\leq c\,.\sup\left\{\left(\sum_{i=1}^{m} | \langle x_i, y\rangle_{\mathcal{H}} |^2\right)^{\frac{1}{2}} : y_i^+ \in \mathcal{H}_+ \| y_i^+ \| \leq 1\right\} \\
&\leq c\,.\sup\left\{\left(\sum_{i=1}^{m} \| x_i \|^2 \| y_i^+ \|^2\right)^{\frac{1}{2}} : \| y_i^+ \| \leq 1\right\}. \\
&\leq c\,.\left\{\left(\sum_{i=1}^{m} \| x_i \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq c\,.\left\{\left(\sum_{i=1}^{m} \| A_{21}x_i^+ \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq c\,.\left\{\left(\sum_{i=1}^{m} \| A_{21} \|^2 \| x_i^+ \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq c\,.\, \| A_{21} \| \left\{\left(\sum_{i=1}^{m} \| x_i^+ \|^2\right)^{\frac{1}{2}}\right\} \\
&\leq \| A_{21} \| .c\,.\left\{\left(\sum_{i=1}^{m} \| x_i^+ \|^2 \| y_i^+ \|^2\right)^{\frac{1}{2}} : y_i^+ \in \mathcal{H}_+\,, y_i^+ = \frac{x_i^+}{\| x_i^+ \|}\right\} \\
&\leq \frac{| \Gamma+\gamma |}{2(Re(\Gamma\overline{\gamma}))^{\frac{1}{2}}} \| A_{21} \| \pi_2.(A).\sup\left\{\left(\sum_{i=1}^{m} | \langle x_i^+, y_i^+\rangle_{\mathcal{H}_+} |^2\right)^{\frac{1}{2}} : \| y_i^+ \| \leq 1\right\}
\end{aligned}$$

Hence A_{21} is absolutely 2-summing with $\pi_2\left(A_{21}\right) \leq \frac{|\Gamma+\gamma|}{2(Re(\Gamma\overline{\gamma}))^{\frac{1}{2}}} \| A_{21} \| \pi_2(A)$. □

The component A_{12} in the matrix representation of A is also absolutely 2-summing. From 4.2.1 and 4.2.2, we have the following theorems.

Theorem 4.1. *Given that the operator A in (4.2) is absolutely 2-summing, each element $A_{jk} \neq 0, i \neq j$ in the matrix representation is absolutely 2-summing for all $A_{jk} = o, j = k$*

Theorem 4.2. *Given that the operator* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_+ \\ \mathcal{K}- \end{pmatrix}$ *is absolutely* 2*-summing, for all* $x_i \in \mathcal{H}, A_{jj} \neq 0$ *and* $A_{jk} = 0$ *where* $j, k = 1, 2$*, then* A_{jj} *is absolutely* 2*-summing.*

The open problem is to work on the case when all components in the matrix representation of A are non zero and absolutely 2-summing given that A is absolutely 2-summing.

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