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AN APPLICATION OF 3-DESIGNS TO CONSTRUCT 3-GDDS OF BLOCK SIZE 4 AND 2 GROUPS

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Abstract. In [3], it was proved that necessary conditions are sufficient for existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 1, 7, 9 \pmod{12}$. It was also proved that for $n \equiv 3 \pmod{12}$ necessary conditions are sufficient except when $\lambda_1 \equiv 9 \pmod{12}$. In this paper, we give the construction of a 3-GDD$(n, 2, 4; 9, 1)$ for $n \equiv 3 \pmod{12}$ and $n \neq 3$. We prove that necessary conditions for the existence of a 3-GDD$(n, 2, 4; 9, 1)$ are sufficient for $n \equiv 3 \pmod{12}$ and $n = 15, 27$. We also prove that necessary conditions for the existence of a 3-GDD$(n, 2, 4; 6s, 8t)$ are sufficient for $n \equiv 10 \pmod{12}$.

Introduction

One of the main problems in combinatorial deign theory is to find construction methods to obtain unknown designs. In [2], a 3-GDD$(n, 2, 4, k, \lambda_1, \lambda_2)$ was defined by extending the definition of a t-design and a group divisible design and gave necessary conditions for its existence. The authors in [2] proved that necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, k, \lambda_1, \lambda_2)$ for $n \equiv 1, 3 \pmod{6}$, except $n \neq 3, 7, 13$ and $\lambda_1 > \lambda_2$. In [3] it was proved that necessary conditions are sufficient for existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 1, 7, 9 \pmod{12}$. The proof relied on the fundamental construction of a 3-GDD$(n, 2, 4, 3, 1)$. In this paper we give the construction of a 3-GDD$(n, 2, 4; 9, 1)$ for $n \equiv 3 \pmod{12}$ and $n \neq 3$. We prove that necessary conditions for the existence of a 3-GDD$(n, 2, 4; 9, 1)$ are sufficient for $n \equiv 3 \pmod{12}$ and $n = 15, 27$. We also prove that necessary conditions for the existence of a 3-GDD$(n, 2, 4; 6s, 8t)$ are sufficient for $n \equiv 10 \pmod{12}$

Preliminaries.

Definition 0.1. [2] A design is a pair $(X, \mathcal{A})$, where X is a set of elements called points, and $\mathcal{A}$ is a collection of nonempty subsets of X called blocks.

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Two identical blocks (repeated blocks) in a design are allowed and hence we refer to $\mathcal{A}$ as a multiset rather than a set of blocks. A design is simple if it does not contain repeated blocks [6]. As an example, let $V = \{1, 2, 3, 4, 5, 6, 7\}$ be a set of 7 elements and $\mathcal{B} = \{\{4, 5, 6, 7\}, \{2, 3, 6, 7\}, \{2, 3, 4, 5\}, \{1, 3, 5, 7\}, \{1, 3, 4, 6\}, \{1, 2, 5, 6\}, \{1, 2, 4, 7\}\}$ be a collection of subsets of V. We note here that there are no repeated blocks. Hence the above design is a simple design.

Definition 0.2. [6, 2] A group divisible design, $GDD(n, m, k; \lambda_1, \lambda_2)$, is a collection of k-element subsets, called blocks, of an nm-set X, where

(i) the elements of X are partitioned into m subsets (called groups) of size n each;
(ii) pairs of distinct elements within the same group are called first associates of each other and appear together in λ_1 blocks and
(iii) any two elements not in the same group are called second associates and appear together in λ_2 blocks.

Example 0.3. A GDD$(3, 2, 4; 3, 1)$ with two groups $\{1, 2, 3\}$ and $\{4, 5, 6\}$ is: $\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5, 6, 1\}, \{4, 5, 6, 2\}, \{4, 5, 6, 3\}\}$.

In the above example we note the following;

(i) Every element occurs in exactly 4 blocks. In any GDD where the groups are of the same size, every element occurs a fixed number of times. This is called the replication number and is denoted by r.
(ii) There are only two groups of the same size that is 3.
(iii) Each block intersects a group in exactly three points or in exactly one point.

One may be able to generalize the concepts of GDDs and t-designs in many ways depending on the structure or restriction on the configurations of the blocks of a GDD. For GDDs with two groups and block size k, the concepts generalize in a natural way:

t-Designs.

Definition 0.4. [6] A t-(v, k, λ) design, or a t-design is a pair (X, B) where X is a v-set of points and B is a collection of k-subsets (blocks) of X with the property that every t-subset of X is contained in exactly λ blocks. The parameter λ is called the index of the design.

Theorem 0.5. *Suppose that $(X, \mathcal{A})$ is a t-(v, k, λ)-deign. Let $Z \subset X, \mid Z \mid = i < t$. Then $(X \setminus Z, \{A \setminus Z : Z \subseteq A \in \mathcal{A}\})$ is a $(t - i)$-$(v - i, k - i, \lambda)$ design.*

It is known that a 5-$(12, 6, 1)$-design exists. Taking cases of $i = 1, 2, 3$, we obtain the following designs: 4-$(11, 5, 1)$, 3-$(10, 4, 1)$, 2-$(9, 3, 1)$.

Definition 0.6. [6] Let v, k and λ be positve integers such that $v > k \geq 2$. A (v, k, λ)- balanced incomplete block design((v, k, λ)-BIBD)is a design $(X, \mathcal{A})$ such that the following properties are satisfied.

(i) $|X| = v$.
(ii) each block contains exactly k points.
(iii) every pair of distinct points is contained exactly λ blocks.

Steiner quadruple systems (SQS)

Definition 0.7. [2] A Steiner quadruple system is an ordered pair (V, B) where V is a finite set of v symbols and B a collection of 4 element subsets of V called blocks (quadruples) with the property that every three element subset of V is a subset of exactly one quadruple B. $|V|$ is called the order of the Steiner quadruple system.

A well known example of a SQS is a SQS (2^n) for $n \geq 3$. An SQS is just a particular example of a t-design. An SQS(n) is also be denoted by 3-$(v, 4, 1)$ and it is known that the necessary conditions are sufficient for existence of 3-$(n, 4, \lambda)$. It is also known that necessary conditions for the existence of an SQS(n) is that $n \equiv 2, 4, \pmod 6$. Hence a 3-$(8, 4, 1)$ and 3-$(10, 4, 1)$ exist.

A classical construction of SQS(2^n). Let V be a set of binary words of length n. Define addition coordinate-wise modulo 2. For example; Let $X = (a_1, a_2, \cdots, a_n)$, $Y = (b_1, b_2, \cdots, b_n)$.
We have $X+Y = ((a_1 + b_1) \pmod 2, (a_2 + b_2) \pmod 2, \cdots, (a_n + a_n) \pmod 2)$. Then (V, B) is a SQS where the set of blocks $B = \{x, y, z, w\}$ where x, y, z and w are 4 distinct binary words of length n such that $x+y+z+w = 0 = (0, 0, 0, \cdots, 0)$. We need to prove that given any triple of distinct words say x, y, z, it is at most one block and second it is in exactly one block. We note that x, y and z are distinct. Let $w = x + y + z$. Then, $x + y + z + w = 0$. So $\{x, y, z, w\}$ may be a block containing the triple and if $x + y + z + w = 0$, then $x + y + z = w$ so the triple will be in at most one block. Note: If x, y, z and w are not all distinct, then no block contains x, y and z and if they are distinct, exactly one block will contain them. Now we must show that in fact x, y, z are distinct. To show that x, y, z are distinct, let $x = w$ then $x + w = 0$. So $x + y + z + w = 0$ becomes $y + z = 0$ and $y = z$ and we have a contradiction. So for every triple is in exactly one quadruple and we have a SQS on 2^n elements.

Theorem 0.8. *If there exists an SQS(n), then there exists an SQS$(2n)$ for $n \geq 4$.*

Proof. We note that, for even n, a resolvable $(n, 2, 1)$-BIBD exists. Let $X$and Y be two groups of cardinality n that is $|X| = |Y| = n, X \cap Y = \phi$. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be resolvable $(n, 2, 1)$-BIBDS having parallel classes, $\Pi_1, \cdot, \cdot, \cdot, \Pi_{n-1}$ and $\psi_1, \cdot, \cdot, \cdot, \psi_{n-1}$ respectively. Also let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be an SQS(n). We define $\varepsilon = \{A \cup B : A \in \Pi_i, \mathcal{B} \in \psi_i, 1 \leq i \leq n - 1\}$. We show that $X \cup Y, C \cup D \cup \varepsilon$ is an SQS(n). Suppose that $\{z_1, z_2, z_3\} \subseteq X \cup Y$. The following cases may arise:

(i) If $|\{z_1, z_2, z_3\}| \cap X = 3$, then, $\{z_1, z_2, z_3\}$ is a subset of unique block in C and is not a subset of any block in $D \cup \varepsilon$.
(ii) If $|\{z_1, z_2, z_3\}| \cap Y = 3$, then, $\{z_1, z_2, z_3\}$ is a subset of unique block in D and is not a subset of any block in $C \cup \varepsilon$.
(iii) Suppose $|\{z_1, z_2, z_3\}| \cap Y = 2$, say, $z_1, z_2 \in X$ and $z_3 \in Y$. There is a parallel unique class say Π_j, such that $\{z_1, z_2\} \in \psi_j$. Then $\{z_1, z_2, z_3\} \subseteq \{z_1, z_2, z_3, z_4\} \in \varepsilon$, and $\{z_1, z_2, z_3\}$ is not a subset of any block in $C \cup D$.

□

We note that there does not exist an SQS(4), because of the restriction that $v > k$ in the definition of t - (v, k, λ)-design. However, for $n = 4$, an SQS($2n$) is SQS(8) and it exists.

Example 0.9. Let $X = \{1, 2, 3, 4, a, b, c, d\}$ be a set of two groups $G_1 = \{1, 2, 3, 4\}$ and $G_2 = \{a, b, c, d\}$, there are 14 blocks of size 4 as shown below:
$\mathcal{A} = \{12ab, 12cd, 34ab, 34cd, 13ac, 13bd, 24ac, 24bs, 14ad, 14bc, 23ad, 23bc, 1234, abcd\}$

The table of necessary conditions where the values of λ_1 and λ_2 are given in Modulo 6 is given [2].

λ_1/λ_2	0	1	2	3	4	5
0	all n	n even	all n	n even	all n	n even
1	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)
2	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4,$ (mod 6)
3	n even	all n	n even	all n	n even	all n
4	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)
5	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)

Table 1

CONSTRUCTION OF 3-GDDS WITH 2 GROUPS AND BLOCK SIZE 4 USING 3-DESIGNS

Construction of a 3-GDD$(n, 2, 4, 6s, 6t)$. For even n, a 3-$(n, 4, 3)$ exists, hence a 3-$(n, 4, 6)$ exists. If $n \equiv 5 \pmod 6$, a 3-$(2n, 4, 6)$ exists, that is 3-GDD$(n, 2, 4, 6, 6)$ exists. To construct the required 3-GDD, we use s copies of a 3-$(2n, 4, 6)$ and $3(t - s)$ copies of 3-GDD$(n, 2, 4, 0, 2)$ for $t - s \geq 0$. When $n \equiv 2, 4, 5 \pmod 6$, we take s copies of 3-$(n, 4, 6)$ together with $3t$ copies of a 3-GDD$(n, 2, 4, 0, 2)$ to get 3-GDD$(n, 2, 4, 6s, 6t)$ which is a required GDD. Hence we have the following theorems.

Theorem 0.10. *Necessary conditions for existence of a* 3*-GDD*$(n, 2, 4, 6s, 6t)$ *for* $n \equiv 2, 4, 5 \pmod 6$ *are sufficient for any positive integers* s *and* t *where* $s \leq t$.

Theorem 0.11. *If a* 3*-GDD* $(n, 2, 4, 6s, 6t)$ *exists for* $n \equiv 2, 4, 5 \pmod 6$*, then, a* 3*-GDD* $(n, 2, 4, \lambda_1 \equiv s \pmod 6, \lambda_2 \equiv t \pmod 6)$ *exists.*

CONSTRUCTION OF 3-GDDS FOR $n = 7, 9, 13$ AND 15

In this section we give fundamental constructions for $n = 7, 9, 13$ and 15 and generalize these constructions to obtain infinitely many 3-GDDs for $n \equiv 1, 3 \pmod 6$ where $\lambda_1 > \lambda_2$ for $\lambda_1 = 6s$ and $\lambda_2 = 6t$ for positive integers s and t.

Construction of 3-GDD$(n, 2, 4, 6s = 12, 6t = 6)$**.** Consider a large set of 7, observe that the set of all 4-subsets of a 7-set is a 3-$(7, 4, 4)$ and hence 3 copies of a 3-$(7, 4, 4)$ on each groups of size 7 together with the blocks of 3 copies of a 3-GDD$(7, 2, 4, 0, 2)$ give the blocks for the required 3-GDD which is 3-GDD$(7, 2, 4, 12, 6)$. Hence we generalize that any positive integer s, the set of all 4-subsets of a $(6s+1)$-set is a 3-$(6s+1, 4, 6s-2)$ and hence $3a$ copies, give us a 3-$(6s+1, 4, 6(3s-1)a)$. It is well known there exists a 3-GDD$(6s+1, 2, 4, 0, 2)$. Taking $3t$ copies of 3-GDD$(6s + 1, 2, 4, 0, 2)$, we get 3-GDD$(6s + 1, 2, 4, 0, 6t)$. Taking the union of the two groups we obtain a 3-GDD$(6s+1, 2, 4, 6(3s-1)a, 6t)$ and hence we have the following theorem.

Theorem 0.12. *A* 3-*GDD*$(6s + 1, 2, 4, \lambda_1 = 6(3s - 1)a, \lambda_2 = 6t)$ *exists for all positive integers* a, s *and* t.

Construction 3-**GDD**$(n, 2, 4, 6s = 12, 6t = 6)$**.** For $n = 9$ we have that the set of all 4-subsets of a 9-set is a 3-$(9, 4, 6)$. If we take 2 copies of a 3-$(9, 4, 6)$ on each groups of size 9 together with the blocks of 3 copies of a 3-GDD$(9, 2, 4, 0, 2)$, we obtain the blocks for the required 3-GDD. From the above, we make the following generalization:

Theorem 0.13. *A* 3-*GDD*$(6s + 3, 2, 4, \lambda_1 = 6(3s - 1)a, \lambda_2 = 6t)$ *exists for all positive integers* a, *and* t.

Construction of 3-GDD$(13, 2, 4, 6s = 12, 6t = 6)$**.** We observe that if $n \equiv 1 \pmod 6$, a 3-$(n, 4, 1)$ exists. If we take $6s$ copies of a 3-$(13, 4, 1)$ together with $3t$ copies of a 3-GDD$(13, 2, 4, 0, 2)$ give a 3-$(n, 2, 4, 12, 6)$ for positive integers s and t and $s \geq 2$ and $t \geq 1$ which is a required 3-GDD.

3-**GDD**$(15, 2, 4, 6s = 12, 6t = 6)$**.** The set of 4-subsets of 15 set is a 3-$(15, 4, 4)$. If we take 3 copies of a 3-$(15, 4, 4)$ together with 3 copies of a 3-GDD$(15, 2, 4, 0, 2)$ on each group, we get the blocks for the required 3-GDD.

$$n \equiv 2, 4, 5 \pmod 6$$

In this section we prove the existence results. It is well known that for $n \equiv 2, 4, 5 \pmod 6$, a 3-GDD$(n, 2, 4, 0, 2)$ and 3-$(2n, 4, 1)$ exist. When λ_1 and λ_2 are of the same parity, we have the following cases: λ_1 and λ_2 are even or λ_1 and λ_2 are odd.

When λ_1 and λ_2 even. let s and t be positive integers and let $\lambda_1 = 2s$ and $\lambda_2 = 2t$. We have two cases: $\lambda_1 > \lambda_2$ and $\lambda_1 < \lambda_2$.

When $\lambda_1 < \lambda_2$. When we take blocks of $2s$ copies of a 3-$(2n, 4, 1)$ that is 3-GDD$(n, 2, 4, 1, 1)$ along with $(t - s)$ copies of a 3-GDD$(n, 2, 4, 0, 2)$ give a 3-GDD$(n, 2, 4, 2s, 2t)$ which is a required 3-GD

When $\lambda_1 > \lambda_2$. For $\lambda_1 > \lambda_2$ and $n \equiv 2, 4, 5 \pmod 6$, a 3-$(n, 4, 2)$ exists. Taking $(s - t)$ copies of a 3-$(n, 4, 2)$ and $2t$ copies of a 3-GDD $(n, 2, 4, 1, 1)$ together, we obtain a 3-GDD $(n, 2, 4, 2s, 2t)$ which is a required 3-GDD.

When λ_1 and λ_2 are odd. We observe that when $\lambda_1 = 2s$ and $\lambda_2 = 2t$, then $\Lambda_1 = 2s-1$ and $\Lambda_2 = 2t-1$ are odd for all positive integers s and t. To construct a 3-GDD$(n, 2, 4, 2s, 2t)$, we take a copy of 3-GDD$(n, 2, 4, 2s-1, 2t-1)$ together with 3- $(2n, 4, 1)$. Hence we have the following lemma.

Lemma 0.14. *A 3-GDD$(n, 2, 4, \lambda_1 = 2s, \lambda_2 = 2t)$ exists for all even λ_1 and λ_2 if and only if a 3-GDD$(n, 2, 4, \Lambda_1, \Lambda_2)$ exists for all odd $\Lambda_1$and Λ_2.*

When λ_1 and λ_2 are of opposite parity

When λ_1 and λ_2 are of the opposite parity, we have the following cases: $\lambda_1 = 2s+1$ and $\lambda_2 = 2t$ or $\lambda_1 = 2s$ and $\lambda_2 = 2t+1$. For $\lambda_1 = 2s+1$ and $\lambda_2 = 2t$, n must be even(that is $n \equiv 2, 4 \pmod 6$). Hence a 3-GDD$(n, 2, 4, 0, 1)$ and a 3-$(2n, 4, 1)$ exist. To construct the required 3-GDD, for $\lambda_1 < \lambda_2$, we use $2s+1$ copies of a 3-$(2n, 4, 1)$ along with $2t-(2s+1)$ of a 3-GDD$(n, 2, 4, 0, 1)$ to obtain a 3-GDD$(n, 2, 4, 2s+1, 2t)$. For the case of $\lambda_1 = 2s$ and $\lambda_2 = 2t+1$, when $\lambda_1 < \lambda_2$ and $n \equiv 2, 4 \pmod 6$, we use $2s$ copies of a 3-$(2n, 4, 1)$ together with $(2t-2s)$ copies of a 3-GDD$(n, 2, 4, 0, 1)$ to get a required 3-GDD. Alternatively we can use $2s$ copies of a 3-$(2n, 4, 1)$ and $(2t+1-2s)$ copies of a 3-GDD$(n, 2, 4, 0, 1)$ on groups G_1 and G_2 to obtain the required 3-GDD. From the above cases, we obtain the following theorem:

Theorem 0.15. *For $n \equiv 2, 4, 5 \pmod 6$, the necessary conditions are sufficient for existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$*

We note here that the above theorem takes care of twelve cells of **table**1 of necessary conditions and has a significant impact on all other cell.

Lemma 0.16. *Necessary conditions are sufficient for existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for even n.*

Construction of a 3-GDD$(n, 2, 4; 0, 1)$

We note first that as $\lambda_1 = 0$, there are no blocks containing three elements from the same group. Secondly, for every pair $\{x, y\}$ of the elements of a groups is in exactly one factor as an edge e. Suppose, e is in 1-factor. The blocks which contain the pair $\{x, y\}$ that is an edge e are precisely $e \cup f_i, i = 1, 2, \cdot, \cdot, \cdot, t$. A triple of elements $\{x, y, z\}$ where z is an elements in G_2 occurs in exactly one block and hence $\lambda_1 = 1$. Example below demonstrates how to construct a 3-GDD$(n, 2, 4; 0, 1)$.

Example 0.17. Let $X = \{1, 2, 3, 4, a, b, c, d\}$ and let $G_1 = \{1, 2, 3, 4\}, G_2 = \{a, b, c\}$. We get the complete graphs K_4 for each group. Below are one factors for each group. Let E and F be sets of one factors for G_1 and G_2 respectively, where, $E = \{E_1, E_2, E_3\}$ and $F = \{F_1, F_2, F_3\}$. $E_1 = \{\{1, 2\}\{3, 4\}\}$, $E_2 = \{\{1, 4\}\{2, 3\}\}$, $E_3 = \{\{1, 3\}\{2, 4\}\}$ and $F_1 = \{\{a, b\}\{c, d\}\}$, $F_2 = \{\{a, d\}\{b, c\}\}$, $F_3 = \{\{a, c\}\{b, d\}\}$. Forming the blocks of $E_i \cup F_j$ of size 4 we get the required 3-GDD. Let us form blocks of $E_i \cup F_j$ of size 4 as follows: $E_1 \cup F_1 = \{1, 2, a, b\}, \{1, 2, c, d\}, \{3, 4, a, b\}\{3, 4, c, d\}$, $E_3 \cup F_2 = \{1, 3, a, d\}, \{1, 3, b, c\}, \{2, 4, a, b\}, \{2, 4, b, c\}$, $E_2 \cup F_3 = \{1, 4, a, c\}, \{1, 4, b, d\}, \{2, 3, a, c\}$,

$\{2,3,b,d\}$. From each block of size 4, we observe that no block that contains three elements from the same group, so $\lambda_1 = 0$ and only one block that contains three elements from different groups. Hence $\lambda_2 = 1$.

Remark 0.18. To demonstrate the construction of a 3-GDD$(n,2,4;0,2)$, we use example 0.17 for a 2- factorizaton of K_4. We note here that each group has only one factor that is for G_1, $F_1 = \{\{1,2\},\{2,3\}\{3,4\}\}$ and for G_2, $F_2 = \{\{a,b\},\{b,c\}\{c,d\}\}$. Forming blocks of $F_1 \cup F_2$ of size 4, we get the following blocks.
$F_1 \cup F_2 = \{1,2,a,b\},\{1,2,b,c\},\{1,2,c,d\},\{1,2,d,a\},\{2,3,a,b\},\{2,3,b,c\},\{2,3,c,d\}$, $\{2,3,d,a\},\{3,4,a,b\},\{3,4,b,c\},\{3,4,c,d\},\{3,4,d,a\},\{4,1,a,b\},\{4,1,b,c\},\{4,1,c,d\}$, $\{4,1,d,a\}$. No triple can be obtained from each quadruple and hence $\lambda_1 = 0$ and there are two triples where one element is obtained from one group and the third element from another group. Hence $\lambda_2 = 2$

Construction of a 3-GDD$(n,2,4;9,1)$ for $n \equiv 3 \pmod{12}$ and $n \neq 3$)

To construct a 3-GDD$(n,2,4;9,1)$ for $n \equiv 3 \pmod{12}$ $n \neq 3$, we use the existing 3-designs and 3-GDDs.

$n = 15$. Here we first establish the existence of existence of $3-(n,4,12)$ and a 3-$(n,2,4;3,1)$ for $n = 15$. We compute the values of replication number r, number of blocks b, the replication number of the first associate pairs r_1 and replication number for the second associate pairs r_2. If all the parameters are integers, we confirm that the design exists. A $3-(15,4,12)$ is a 3-GDD$(15,2,4;12,0)$. After computing using expressions of necessary conditions, $b = 2,730, r = 364$, r_1 and r_2 are integers. Hence A 3-GDD$(15,2,4;12,0)$ exists. It is also known in [3] that a 3-GDD$(n,2,4;,3,1)$ exists. For $n = 15$, we obtain a 3-GDD$(15,2,4;3,1)$. This exists with r, b, r_1 and r_2 as integers. To Construct a 3-GDD$(15,2,4;9,1)$, we take a copy of a 3-GDD$(15,2,4;3,1)$ together with $\frac{1}{2}$ copies of a 3-GDD$(15,2,4;12,0)$ that is a 3-GDD$(15,2,4;6,0)$ on each group to get a required 3-GDD$(15,2,4;3,1)$.

Lemma 0.19. *Necessary conditions for existence of a* 3-*GDD*$(n,2,4;9,1)$ *are sufficient for* $n = 15$.

n= 27. For $n = 27$, a 3-GDD$(27,2,4;3,1)$ and a $3-(27,4,12)$ exist. Taking a copy of a 3-GDD$(n,2,4;3,1)$ for $n = 27$ on each group, together with $\frac{1}{2}$ copies of a $3-(27,4,12)$ to get the required 3-GDD$(27,2,4;9,1)$. This result together with lemma 0.19 we obtain the following theorem.

Theorem 0.20. *Necessary conditions for existence of a* 3-*GDD*$(n,2,4;\lambda_1 = 9,\lambda_2 = 1)$ *are sufficient for* $n \equiv 3 \pmod{6}$ *and* $n = 15, 27$.

Necessary conditions on n for $\lambda_1, \lambda_2 \geq 6 \pmod{12}$

To study conditions on n for values of λ_1 and λ_2 computed in modulo 12 where $\lambda_1 \geq 6$ and $\lambda_2 \geq 6$, the following table of necessary conditions was obtained.

λ_1/λ_2	6	7	8	9	10	11
6	all n	n even	all n	n even	all n	n even
7	$2,4,8,10$ (mod 12)	$1,2,4,8,10,$ 11 (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,5,7,$ $8,10,11$ (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,5,7,$ $8,10,11$ (mod 12)
8	$1,2,4,5,7,$ $10,11$ (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,7,$ $10,11$ (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,5,7,$ $8,10,11$ (mod 12)	$2,4,8,10$ (mod 6)
9	n even	all n	n even	all n	n even	all n
10	$1,2,4,5,7$ $10,11$ (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,5,7$ $8,10,11$ (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,5,7,$ $8,10,11$ (mod 12)	$2,4,8,10$ (mod 12)
11	$2,4,8,10$ (mod 12)	$1,2,4,5,7,$ $8,10,11$ (mod 12)	$2,4,8,10$ (mod 12)	$1,2,4,5,7,$ $8,10,11$ (mod 12)	$2,4,8,10$ (mod 6)	$1,2,4,7,$ $8,10,11$ (mod 6)

Table 2

From table 2 of necessary conditions, we note the following results: A 3-GDD$(n,2,4,6,7)$, a 3-GDD$(n,2,4,6,9)$ and a 3-GDD$(n,2,4,6,11)$ exist for even positive integers n. We note here that in all cases above, λ_1 and λ_2 are of opposite parity and $\lambda_1 \leq \lambda_2$. Hence we have the following lemma.

Lemma 0.21. *For n even, a 3-GDD$(n,2,4,\lambda_1 = 2t, \lambda_2 = 2s+1)$ exists for all positive integers s and t.*

We also note that for n even, the following 3-GDDs exist. Viz: A 3-GDD$(n,2,4;9,6)$,a 3-GDD$(n,2,4;9,8)$ and a 3-$(n,2,4;9,10)$. Hence we have the following result.

Lemma 0.22. *A 3-GDD$(n,2,4;\lambda_1 = 2t+1, \lambda_2 = 2s)$ exists for even positive integers n.*

Remark 0.23. From lemma 0.21 $\lambda_1 \leq \lambda_2$ and in lemma 0.22 $\lambda_1 \geq \lambda_2$.

Theorem 0.24. *A necessary condition for a 3-GDD$(n,2,4;\lambda_1,\lambda_2)$ to exist when λ_1 and λ_2 are of opposite parity is that n must be even.*

As an application to theorem 0.24, a 3-GDD$(8,2,4;9,6)$ exists and a 3-GDD$(11,2,4;6,9)$ does not exist. A 3-GDD$(n,2,4;6,6)$, a 3-GDD$(n,2,4;6,8)$ and a 3-GDD(n,2,4;6,10) exist for all positive integers n.

Lemma 0.25. *A 3-GDD$(n,2,4;\lambda_1 = 2t, \lambda_2 = 2s)$ exists for all positive integers n, s and t.*

Lemma 0.26. *A 3-GDD$(n,2,4;\lambda_1 = 2t+1, \lambda_2 = 2s+1)$ exists for all positive integers n.*

As an application of lemmas 0.25 and 0.26, a 3-GDD$(n,2,4;6,10)$ and a 3-GDD$(n,2,4;9,7)$ exist.

Theorem 0.27. *For all positive integers n, a necessary condition for the existence of a 3-GDD$(n,2,4;\lambda_1,\lambda_2)$ is that λ_1 and λ_2 are of the same parity.*

Theorem 0.28. *Necessary conditions are sufficient for the existence of a* 3-$GDD(n, 2, 4; 6s, 8t)$ *for all positive integers* n, s *and* t.

Theorem 0.29. *For* $n \equiv 1, 11 \pmod{12}$, *a* 3-$GDD(n, 2, 4; 2t, 2s)$ *and a* 3-$GDD(n, 2, 4, 2t+1, 2s+1)$ *exist.*

Construction of a 3-GDD$(n, 2, 4; 6s, 8t)$ for $n \equiv 8, 10 \pmod{12}$

It is known that a 3-$(n, 4, 1)$ that is $SQS(n)$ and 3- $(10, 4, 1)$ exist. A 3-GDD$(n, 2, 4; 0, 2)$ exists for all positive integers n. Hence a 3-GDD$(10, 2, 4; 0, 2)$ exists. To construct the required GDD, we use $6s$ copies of a 3-$(n, 4, 1)$ on G_1 and G_2 together with $4t$ copies of a 3-GDD$(n, 2, 4; 0, 2)$ to get a 3-GDD$(n, 2, 4; 6s, 8t)$ which is a required GDD. Hence we have:

Theorem 0.30. *Necessary conditions for the existence of a* 3-$GDD(n, 2, 4; 6s, 8t)$ *are sufficient for* $n \equiv 10 \pmod{12}$.

If $SQS(n)$ exists, a $SQS(2n)$ also exists and hence a 3-$(2n, 4, 1)$ exists for $n \geq 4$. To construct a 3-GDD$(8, 2, 4; 6s, 8t)$, we take $6s$ copies of 3-$(2n, 4, 1)$ on each group together with $8t - 6s$ copies a 3-GDD$(n, 2, 4; 0, 1)$ to get a required GDD. Hence necessary conditions for existence of 3-$(8, 2, 4; 6s, 6t)$ are sufficient. Using the fact if an $SQS(n)$ exists, then an $SQS(2n)$ exist, we have that if 3-$(8, 4, 1)$ and a 3-$(10, 4, 1)$ exist, a 3-$(16, 4, 1)$ and a 3-$(20, 4, 1)$ exist.

Construction of a 3-GDD$(n, 2, 4; 6, 7+2s)$. To construct a 3-GDD$(n, 2, 4; 6, 7+2s)$, we use the existing 3-designs and 3-GDDs as follows: It is known that for even n, a $3 - (2n, 4, 1)$ and a 3-GDD$(n, 2, 4; 0, 1)$ exist. A $3 - (2n, 4, 1)$ is a 3-GDD$(n, 2, 4; 1, 1)$. Taking a copy of a 3-GDD$n, 2, 4; 0, 1$ together with 6 copies on each group of a 3-GDD$(n, 2, 4; 1, 1)$ gives a 3-GDD$(n, 2, 4; 6, 7)$ which is the required 3-GDD. We can also take 6 copies of a 3-GDD$(n, 2, 4; 1, 1)$ on both groups together with 3 and 5 copies of a $3 - (n, 2, 4, 0, 1)$ to get a 3-GDD$(n, 2, 4; 6, 9)$ and a 3-GDD$(n, 2, 4; 6, 11)$. We make a generalisation and obtain the following theorem.

Theorem 0.31. *Necessary conditions are sufficient for the existence of a* 3-$GDD(n, 2, 4; 6, 7 + 2s)$ *for odd integer* s *and even* n.

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