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REPRESENTATIONS OF $SU(2)$ AND SOME APPLICATIONS

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Abstract. We compute the ring of periodic complex K-theory of the classifying space of $SU(2)$ by the Atiyah Segal completion theorem and we compute its ring of connective K-theory by investing the Atiyah Hirzebruch spectral sequence and a cofibration spectra map connecting the connective K-theory and the integral cohomology theory.

1. Introduction

It is well known that the determination of the irreducible representations of certain compact Lie groups allow to solve the heat equations [7] . On the other hand, in the 60's, Atiyah defined in [2], for any paracompact space the ring of topological K-theory.
Thanks to the work of Atiyah-Segal in [3], which allows us the computation of the topological K-theory ring of the classifying space of compact Lie groups by completing the representation ring generated by their irreducible representations with respect to the augmentation ideal.

In this work we focus on the special unitary group $SU(2)$ and its classifying space $BSU(2)$.
In section 2, we determine the ring of complex representations $R(SU(2))$ and its completion $\widehat{R}(SU(2))$. We prove that $R(SU(2)) = \mathbb{Z}[v]$ where $v = t - 2$ and t being the stantard representation.
In section 3, using the Atiyah-Segal completion theorem, we deduce the ring of the K-theory of $BSU(2)$, $KBSU(2) = \mathbb{Z}[[w]]$ where w is the vector bundle associated to the representation v. We conclude by determining the ring of the connective K-theory of $BSU(2)$.
As was mentionned in [6] knowing connective K-theory of BSU(2) sheds light on his cobordism.

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2. The representation ring $R(SU(2))$ and its completion

The special unitary group

$$\mathrm{SU}(2)=\left\{\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}: \quad \alpha,\beta\in\mathbb{C}, |\alpha|^2+|\beta|^2=1\right\} \tag{2.1}$$

Where the overline denotes complex conjugation.
It is a simply-connected, compact Lie group diffeomorphic to the 3-sphere S^3.
In [7], Faraut determines the irreducible representations of SU(2), Indeed :
Every irreducible complex representation of $SU(2)$ is equivalent to $(\pi_n, \mathcal{P}_n)$ where

$$\begin{array}{cccc}\pi_n: & SU(2) & \longrightarrow & Aut_{\mathbb{C}}(\mathcal{P}_n)\\ & g=\begin{bmatrix}a & b\\ c & d\end{bmatrix} & \longmapsto & \pi_n(g)(f)(x,y)=f(ax+cy,bx+dy)\end{array} \tag{2.2}$$

where $\mathcal{P}_n$ is the complex vector space of homogeneous polynomials in two variables with complex coefficient of degree n.
The representations π_n verify the Clebsh-Gordan relations [7]

$$\pi_p\otimes\pi_q=\sum_{k=0}^{min(p,q)}\pi_{p+q-2k} \tag{2.3}$$

Let denote by t the class of the standard representation π_1 in $R(SU(2))$.

Theorem 2.1.

$$R(SU(2))=\mathbb{Z}[t] \tag{2.4}$$

where $|t|=2$.

Proof. By the Clebsh-Gordan relations, we have for all integers $n\geqslant 1$.

$$\pi_1\otimes\pi_n=\sum_{k=0}^{min(n,1)}\pi_{n+1-2k}=\pi_{n+1}\oplus\pi_{n-1} \tag{2.5}$$

Therefore for any integer $n\geqslant 1$:

$$\pi_{n+1}=\pi_1\pi_n-\pi_{n-1}. \tag{2.6}$$

Let show for any integer $n\geqslant 0$, that $\pi_n\in\mathbb{Z}[t]$.
We have $\pi_0=P(\pi_1)$ where $P(X)=1$ and $\pi_1=P(\pi_1)$ where $P(X)=X$.
Let's assume that for $n\geqslant 1$, π_n and π_{n-1} belong to $\mathbb{Z}[t]$.
By Equation (2.6), we have $\pi_{n+1}\in\mathbb{Z}[t]$.
Thus, by induction for all $n\geqslant 0$, $\pi_n\in\mathbb{Z}[t]$. Therefore $R(SU(2))=\mathbb{Z}[t]$.

□

Theorem 2.2. *The completion ring of $R(SU(2))$ with respect to its augmentation ideal $I(SU(2))$ is :*

$$\widehat{R}(SU(2))=\mathbb{Z}[[v]] \tag{2.7}$$

where $v=t-2$.

Proof. We have $\mathbb{Z}[v] = \mathbb{Z}[t]$ where $v = t - 2$.
Then $R(SU(2)) = \mathbb{Z}[v]$ and the augmentation ideal $I(SU(2))$ is generated by v. Hence

$$\widehat{R}(SU(2)) = \varprojlim \frac{R(SU(2))}{I^n} = \varprojlim \frac{\mathbb{Z}[t]}{(t-2)^n} = \varprojlim \frac{\mathbb{Z}[v]}{(v)^n} = \mathbb{Z}[[v]].$$

□

3. The ring $K^*(BSU(2))$

In [3] Atiyah and Segal describe the K-theory of the classifying space BG for any compact Lie group G.
Let V be a finite dimensional complex representation of G, then we can form the complex vector bundle over BG with fiber V.

$$(EG \times V)/G = EG \times_G V \longrightarrow BG \tag{3.1}$$

Where G acts on the right on the universal space EG and on the left on V.
So we have a naturel map

$$R(G) \longrightarrow K^0(BG) \tag{3.2}$$

Let $I(G)$ be the kernel of the augmentation map $R(G) \longrightarrow \mathbb{Z}$ sending a representation to its dimension and $\widehat{R}(G) = \varprojlim \dfrac{R(G)}{I(G)^n}$ be the $I(G)$-adic completion of $R(G)$. Then the Atiyah Segal completion theorem asserts that the map (3.2) induces an isomorphism $\widehat{R}(G)) \cong K^0(BG)$ and $K^1(BG) = 0$.

Theorem 3.1.

$$K^0(BSU(2)) = \mathbb{Z}[[w]] \tag{3.3}$$

Moreover,

$$K^1(BSU(2)) = 0. \tag{3.4}$$

Proof. Let $W_{SU(2)} = (ESU(2), p, BSU(2))$ be the universal fiber bundle of $SU(2)$. For every finite complex representation φ of $SU(2)$ we associated the vector bundle $W_{SU(2)}(\varphi)$ over $BSU(2)$.
Since $SU(2)$ is a compact Lie group . The Atiyah augmented ring homomorphism

$$\begin{array}{rccc} \mathcal{A}\colon & R(SU(2)) & \longrightarrow & K^*(BSU(2)) \\ & \sum_i n_i t^i & \longmapsto & \sum_i n_i w^i \end{array} \tag{3.5}$$

where $\mathcal{A}(t) = W_{SU(2)}(t) = w$.
Induces after completion the augmented filtered complete ring isomorphism.

$$\widehat{\mathcal{A}}\colon \widehat{R}(SU(2))) \longrightarrow K^*(BSU(2)) \tag{3.6}$$

Hence we deduce that $K^0(BSU(2)) = \widehat{\mathcal{A}}(R(SU(2))) = \widehat{\mathcal{A}}(\mathbb{Z}[[v]]) = \mathbb{Z}[[w]]$ and $K^1(BSU(2)) = 0$.

□

4. The ring $k^*(BSU(2))$

The complex K-theory is represented by a 2-periodic ring Ω-spectrum KU where $KU_{2n} = BU \times \mathbb{Z}$ and $KU_{2n+1} = U$, by Bott periodicity theorem the homotopy groups of KU are

$$\pi_i(KU) = \begin{cases} \mathbb{Z}\{u^i\} & if \quad \text{n is even} \\ 0 & if \quad \text{n is odd} \end{cases} \tag{4.1}$$

Where u is the Bott element of degree 2.
Hence its coefficient ring is the graded ring $\pi_*(KU) = \mathbb{Z}[u, u^{-1}]$.
We can form the connective cover of this ring specrtum denoted by ku, where $ku_n = KU(n, ..., \infty)$ is an nth connected cover of the space KU_n for $n \geq 0$ and $ku_{2n} = BU,\ ku_{2n+1} = U$ for $n \leqslant 0$.
The morphisms adjoint to the structure morphisms are the maps see([1]).

$$\begin{cases} ku_{2n+2} = BU(2n+1, ..., \infty) & \longrightarrow & ku_{2n} = BU(2n, ..., \infty) \\ ku_{2n+1} = U(2n+1, ..., \infty) & \longrightarrow & ku_{2n-1} = U(2n-1, ..., \infty) \end{cases} \tag{4.2}$$

Hence we obtain a ring spectrum map $ku \longrightarrow KU$ which induces the isomorphisms on homotopy groups in positive degree. Hence

$$\pi_i(ku) = \begin{cases} \pi_i(KU) & if \quad i \geq 0 \\ 0 & if \quad i < 0 \end{cases} \tag{4.3}$$

As a graded ring $\pi_*(ku) = \mathbb{Z}[u]$.
We call the corresponding cohomology theory the reduced connective K-Theory and we have $ku^n(X) = [(X, x_0), (ku_n, *)]$ for all pointed CW-complex (X, x_0).
The main theorem is:

Theorem 4.1. *The connective ring of the classifying space $BSU(2)$ is*

$$ku^*(BSU(2)) = \mathbb{Z}[u][[c_2]] \tag{4.4}$$

where $c_2 = c_2(\pi_1)$ is the second Chern class of the standard representation π_1.

The proof of the main theorem requires some technical lemmas:

1) The maps (4.2) induce a morphism see([1]) $\beta\colon (k_n) \longrightarrow (k_n)$ of Ω-spectra. Hence β is the morphism of spectra corresponding to multiplication by u in ku-cohomology.
2) The maps see([1])

$$\begin{cases} k_{2n} = BU(2n, ..., \infty) & \longrightarrow & k_{2n} = (H\mathbb{Z})_{2n} = BU(2n) \\ k_{2n+1} = U(2n+1, ..., \infty) & \longrightarrow & (H\mathbb{Z})_{2n+1} = U(2n+1) \end{cases} \tag{4.5}$$

induce a ring homomorphism $\eta\colon (k_n) \longrightarrow (H\mathbb{Z})$ of Ω-spectra where $H\mathbb{Z}$ is the spectrum associated to the integral cohomology theory. So that it induces the homomorphism $\eta^*\colon ku^*(pt) \to H^*(pt; \mathbb{Z})$ given by $\eta^i = 0$ if $i \geq 0$ and the identity if $i = 0$.

Adams in [1] showed the following result.

Lemma 4.2. [8] *The sequence $(k_n) \xrightarrow{\beta} (k_n) \xrightarrow{\eta} (H\mathbb{Z})$ is a cofibration sequence of spectra.*

Therefore it gives the long exact sequence of $\mathbb{Z}[u]$-modules.

Corollary 4.3. *For every $n \geq 2$. We have a long exact sequence of $\mathbb{Z}[u]$-modules:*

$$\begin{tikzcd} \dots \arrow[r] & \tilde{H}^{n-3}(BSU(2),\mathbb{Z}) \arrow[r, "\delta"] & k^n(BSU(2)) \arrow[r, "\beta^n"] & k^{n-2}(BSU(2)) \arrow[r, "\eta^{n-2}"] & \tilde{H}^{n-2}(BSU(2),\mathbb{Z}) \arrow[d, "\delta"] \\ & & & & k^{n+1}(BSU(2)) \arrow[d] \\ & & & & \vdots \end{tikzcd} \tag{4.6}$$

Where δ is the boundary homomorphism.

Next we establish that the integral cohomology $H^*(BSU(2),\mathbb{Z})$ is concentrated in even degrees.

Lemma 4.4. *The integral cohomology of $BSU(2)$*

$$H^*(BSU(2),\mathbb{Z}) = \mathbb{Z}[c_2]. \tag{4.7}$$

Proof. The integral cohomology of $BU(2)$ is $H^*(BU(2),\mathbb{Z}) = \mathbb{Z}[c_1, c_2]$ where $c_1 \in H^2(BU(2),\mathbb{Z})$ and $c_2 \in H^4(BU(2),\mathbb{Z})$ are the first and second universal Chern class.
The exact sequence of groups $1 \to SU(2) \xrightarrow{j} U(2) \xrightarrow{det} U(1) \to 1$ yields the fibration $BSU(2) \xrightarrow{Bj} BU(2) \xrightarrow{Bdet} BU(1)$, by Serre spectral sequence the homomorphism $Bj^*\colon H^*(BU(2),\mathbb{Z}) \to H^*(BSU(2),\mathbb{Z})$ is an epimorphism and $j^*(c_1) = 0$, it follows that its restriction to $\mathbb{Z}[c_2]$ is an isomorphism. □

Proof of Theorem 4.1. Since $H^*(BSU(2),\mathbb{Z})$ is concentraded in even degrees the Atiyah-Hirzebruch spectral sequence $H^*(BSU(2), ku^*) \Longrightarrow ku^*(BSU(2))$ is concentraded in even degree, so is $ku^*(BSU(2))$ and the boundary homomorphism $\delta\colon H^n(BSU(2),\mathbb{Z}) \to ku^{n+3}(BSU(2))$ is therefore 0, hence $\beta^n\colon ku^n(BSU(2)) \to ku^{n-2}(BSU(2))$ is a monomorphism .
Then $ku^*(BSU(2))$ is the complete $\mathbb{Z}[u]$-algebra freely generated by c_2.

Remark 4.5. The techniques used above by the authors to calculate the K-theory and the connective K-theory of BG for G a compact Lie group inspired K. Azi and H.Hamraoui to generalize them in order to find the K-toplogical theory of the classifying space $BSL_d(K)$, K a commutative field and $BSL_d(K)$ the special group of order d. Indeed in [4], K. Azi and H. Hamraoui determined the representations of SL_d and then in [5] they proved an analogue of the Atiyah-Segal completion theorem in the homotopy category of schemas. This allowed them to determine the K- topological theory of $BSL_d(K)$.

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References

1. J.F Adams, *"On Chern characters and the structure of the unitary group"*, Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 57, No. 2, Cambridge University Press, 1961.
2. M.F.Atiyah ,*"K-theory"*, ©W.A.BENJAMIN ,Inc,New York , 1967
3. M.F.Atiyah, and G.B Segal,*"Equivariant K-theory and completion"*, Journal of Differential Geometry 3.1-2 (1969): 1-18.
4. K.Azi and H.Hamraoui, *"The representation ring of SL_d"*, Gulf Journal of Mathematics, 4(2016), Issue 4, 48-53
5. K.Azi and H.Hamraoui, *"Atiyah motivic theorem for the special group SL_d"*, Gulf Journal of Mathematics, 4 (2016), Issue 4, 54-59.
6. R.R.Bruner, and J.P.C Greenlees.*"Connective Real K-Theory of Finite Groups"*, No. 169, American Mathematical Soc, 2010.
7. J. Faraut,*"Analysis on Lie groups"*, Cambridge, 2008.
8. L.Magalhes,*"Some Results in the Connective K-Theory of Lie Groups"*, Canadian Mathematical Bulletin 31.2 (1988): 194-199.
9. G.Segal,*"Equivariant K-theory"*, Publications mathématiques de l'IHÉS 34.1 (1968): 129-151.

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