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COMMON FIXED POINT THEOREMS IN S-METRIC SPACES USING $(E.A)$-PROPERTY AND AN IMPLICIT RELATION

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Abstract. In this paper, we prove some common fixed point theorems in S-metric spaces using $(E.A)$-property for two pairs of self-mappings via an implicit relation and give an example to support the result. The results presented in this paper generalize, extend and enrich various results in the existing literature.

1. Introduction

The Banach contraction principle [4] is the most celebrated fixed point theorem and has been generalized in various directions. The famous Banach contraction principle states that every self mapping F defined on a complete metric space (X, ρ) satisfying the condition:

$$\rho(F(x), F(y)) \leq k\, \rho(x, y), \tag{1.1}$$

for all $x, y \in X$, where $k \in (0, 1)$, has a unique fixed point and for every $x_0 \in X$, a sequence $\{F^n x_0\}_{n\geq 1}$ is convergent to the fixed point.

When Banach formulated the notion of contraction and proved the famous theorem, scientist and mathematicians around the world are publishing new results that associated either to establish a generalization of metric space or to get an improvement of contractive condition. In the literature most famous improvement of contractive condition is certainly the papers of Edelstein-Nemytskii, Boyd-Wong, Meir-Keeler, Kannan, Chatterjae, Zamfirescu and Ciric. In addition to the improvement of Banach contractive conditions, more and more attention is devoted itself to the generalization of metric spaces, such as 2-metric spaces, D^*-metric spaces, partial metric spaces, b-metric spaces, cone metric spaces, G-metric spaces and others.

Mustafa and Sims [12] introduced a new notion of generalized metric space called G-metric space and gave a modification to the Banach contraction principle. One of the generalizations of the metric space is given by Sedghi *et al.* [27].

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Sedghi *et al.* [27] introduced the concept of S-metric spaces which generalized G-metric spaces and D^*-metric spaces. In [27] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in the setting of S-metric spaces for a self-map.

Gupta [7] in 2013, introduced the notion of cyclic contraction in S-metric spaces and proved some fixed theorems which are proper generalizations of the results of Sedghi *et al.* [27].

Generalizing the Banach contraction principle, Jungck [9] initiated the study of common fixed point for a pair of commuting mappings satisfying contractive type conditions. In 1982, Sessa [29] introduced a weaker concept of commutativity, which is generally known as weak commutativity and proved some interesting results on the existence of common fixed points for a pair of self maps. He also showed that weak commuting mappings are commuting but the converse need not to be true. Later, Jungck [10] generalized the concept of weak commutativity by introducing the notion of compatible mappings which is more general than weakly commuting mappings and showed that weak commuting maps are compatible but converse need not be true. In 1996, Jungck [11] generalized the concept of compatibility by introducing weakly compatible mappings.

The study of common fixed points for non compatible mappings was initiated by Pant [14]. In 2002, Aamri and El Moutawakil [1] introduced a new concept called $(E.A)$-property for pair of mappings which is a generalization of non compatible mappings and they proved some common fixed point theorems. The concept of $(E.A)$-property allows us to replace the completeness requirement of the space by a more general condition of closeness of range. In [16], Pathak *et al.* established a common fixed point theorem in metric space for an integral type condition and using implicit relation and the $(E.A)$ property. Some authors showed that the notion of weakly compatible mappings and mappings satisfying $(E.A)$-property are independent (see, [15], [17]).

Many classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [17], [18] and in some other papers.

This direction of research produced a consistent literature on fixed point, common fixed point and coincidence point theorems in various ambient spaces. For more details see [3, 5, 6, 8, 19, 20, 22, 23, 24, 25, 26].

In 2016, Tiwari and Gupta [30] proved some common fixed point theorems in metric spaces satisfying an implicit relation involving quadratic terms. In 2019, Neog *et al.* [13] prove common fixed results of set valued maps for A_φ-contraction and generalized ϕ-type weak contraction in the setting of metric spaces.

Recently, Tiwari and Thakur in [31] proved some common fixed point theorems for pair of mappings satisfying common $(E.A)$-property in the setting of complete metric spaces and give application of the established result.

In the present paper, we prove some common fixed point theorems on S-metric spaces using $(E.A)$-property via an implicit relation and give an example to support the result. Our results generalize, extend and enrich several results from the existing literature.

2. Preliminaries

In this section, we recall some definitions and lemmas to prove our main results.

Definition 2.1. ([27]) Let X be a nonempty set and let $\mathcal{S}\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all u, v, w, $t \in X$:

$(S1)$ $\mathcal{S}(u,v,w) = 0$ if and only if $u = v = w$;

$(S2)$ $\mathcal{S}(u,v,w) \leq S(u,u,t) + S(v,v,t) + S(w,w,t)$.

Then the function $\mathcal{S}$ is called an $\mathcal{S}$-metric on X and the pair $(X,\mathcal{S})$ is called an $\mathcal{S}$-metric space or simply SMS.

Example 2.2. ([27]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(u,v,w) = \|v + w - 2u\| + \|v - w\|$ is an $\mathcal{S}$-metric on X. In general, if X is a vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm on X. Then it is easy to see that

$$\mathcal{S}(u,v,w) = \|\alpha\, v + \beta\, w - \lambda\, u\| + \|v - w\|,$$

where $\alpha + \beta = \lambda$ for every $\alpha, \beta \geq 1$, is an $\mathcal{S}$-metric on X.

Example 2.3. ([27]) Let X be a nonempty set and d be an ordinary metric on X. Then $\mathcal{S}(u,v,w) = d(u,w) + d(v,w)$ for all $u,v,w \in X$ is an $\mathcal{S}$-metric on X.

Example 2.4. ([28]) Let $X = \mathbb{R}$ be the real line. Then $\mathcal{S}(u,v,w) = |u - w| + |v - w|$ for all $u,v,w \in \mathbb{R}$ is an $\mathcal{S}$-metric on X. This $\mathcal{S}$-metric on X is called the usual $\mathcal{S}$-metric on X.

Definition 2.5. ([28]) Let $(X,\mathcal{S})$ be an $\mathcal{S}$-metric space. For $r > 0$ and $u \in X$ we define the open ball $B_{\mathcal{S}}(u,r)$ and closed ball $B_{\mathcal{S}}[u,r]$ with center u and radius r as follows, respectively:

$$B_{\mathcal{S}}(u,r) = \{v \in X : \mathcal{S}(v,v,u) < r\},$$

$$B_{\mathcal{S}}[u,r] = \{v \in X : \mathcal{S}(v,v,u) \leq r\}.$$

Example 2.6. ([28]) Let $X = \mathbb{R}$. Denote by $\mathcal{S}(u,v,w) = |v + w - 2u| + |v - w|$ for all $u,v,w \in \mathbb{R}$. Then

$$\begin{aligned} B_{\mathcal{S}}(1,2) &= \{v \in \mathbb{R} : \mathcal{S}(v,v,1) < 2\} = \{v \in \mathbb{R} : |v-1| < 1\} \\ &= \{v \in \mathbb{R} : 0 < v < 2\} = (0,2), \end{aligned}$$

and

$$\begin{aligned} B_{\mathcal{S}}[2,4] &= \{v \in \mathbb{R} : S(v,v,2) \leq 4\} = \{v \in \mathbb{R} : |v-2| \leq 2\} \\ &= \{v \in \mathbb{R} : 0 \leq v \leq 4\} = [0,4]. \end{aligned}$$

Definition 2.7. ([27], [28]) Let $(X,\mathcal{S})$ be an $\mathcal{S}$-metric space and $A \subset X$.

$(A1)$ The subset A is said to be an open subset of X, if for every $u \in A$ there exists $r > 0$ such that $B_{\mathcal{S}}(u,r) \subset A$.

$(A2)$ A sequence $\{u_n\}$ in X converges to $u \in X$ if $S(u_n,u_n,u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(u_n,u_n,u) < \varepsilon$. We denote this by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.

$(A3)$ A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(u_n, u_n, u_m) < \varepsilon$.

$(A4)$ The $\mathcal{S}$-metric space $(X, \mathcal{S})$ is called complete if every Cauchy sequence in X is convergent.

$(A5)$ Let τ be the set of all $A \subset X$ having the property that for every $u \in A$, A contains an open ball centered at u. Then τ is a topology on X (induced by the $\mathcal{S}$-metric space).

$(A6)$ A nonempty subset A of X is $\mathcal{S}$-closed if closure of A coincides with A.

Definition 2.8. ([27]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \leq k < 1$ such that

$$\mathcal{S}(Tu, Tv, Tw) \leq k\, S(u, v, w), \tag{2.1}$$

for all $u, v, w \in X$.

Remark 2.9. ([27]) If the $\mathcal{S}$-metric space $(X, \mathcal{S})$ is complete and $T\colon X \to X$ is a contraction mapping, then T has a unique fixed point in X (see [27], Theorem 3.1).

Definition 2.10. ([27]) Let $(X, \mathcal{S})$ and $(X', \mathcal{S}')$ be two $\mathcal{S}$-metric spaces. A function $g\colon X \to X'$ is said to be continuous at a point $y_0 \in X$ if for every sequence $\{y_n\}$ in E with $\mathcal{S}(y_n, y_n, y_0) \to 0$, $\mathcal{S}'(g(y_n), g(y_n), g(y_0)) \to 0$ as $n \to \infty$. We say that g is continuous on X if g is continuous at every point $y_0 \in X$.

Definition 2.11. Let X be a non-empty set and let $f, g\colon X \to X$ be two self mappings of X. Then a point $z \in X$ is called a

$(\boldsymbol{\Theta_1})$ fixed point of operator f if $f(z) = z$;

$(\boldsymbol{\Theta_2})$ common fixed point of f and g if $f(z) = g(z) = z$.

Definition 2.12. ([2]) Let f and g be single valued self-mappings on a set X. If $u = fz = gz$ for some $z \in X$, then z is called a coincidence point point of f and g, and u is called a point of coincidence of f and g.

Definition 2.13. ([10]) Let $\mathcal{R}$ and $\mathcal{T}$ be single valued self-mappings on a set X. Mappings $\mathcal{R}$ and $\mathcal{T}$ are said to be commuting if $\mathcal{R}\mathcal{T}\alpha = \mathcal{T}\mathcal{R}\alpha$ for all $\alpha \in X$.

Definition 2.14. ([11]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X. Mappings $\mathcal{A}$ and $\mathcal{B}$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{A}u = \mathcal{B}u$ for some $u \in X$ implies $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u$.

Definition 2.15. ([1]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and let $\mathcal{P}, \mathcal{R}\colon X \to X$ be two self mappings of X. The pair $(\mathcal{P}, \mathcal{R})$ is said to satisfies the $(E.A)$-property if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} \mathcal{P}y_n = \lim_{n\to\infty} \mathcal{R}y_n = v$ for some $v \in X$.

Example 2.16. Let $X = [0, 1]$ and let $\mathcal{P}, \mathcal{R}\colon X \to X$ be defined by $\mathcal{P}(x) = \frac{x}{2}$ and $\mathcal{R}(x) = \frac{x}{4}$. Define the function $\mathcal{S}\colon X^3 \to [0, \infty)$ by

$$\mathcal{S}(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$, then $\mathcal{S}$ is an $\mathcal{S}$-metric on X. Now, we show that the pair $(\mathcal{P}, \mathcal{R})$ satisfies the $(E.A)$ property. For this, consider the sequence $\{y_n\} = \{\frac{1}{2n+1}\}_{n\geq 1}$. Clearly $\{y_n\}$ is in X and note that $\mathcal{P}y_n = \frac{y_n}{2} = \frac{1}{2(2n+1)}$ and $\mathcal{R}y_n = \frac{y_n}{4} = \frac{1}{4(2n+1)}$ for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned}\mathcal{S}(\mathcal{P}y_n, \mathcal{P}y_n, 0) = \mathcal{S}\Big(\frac{1}{2(2n+1)}, \frac{1}{2(2n+1)}, 0\Big) &= \max\Big\{\frac{1}{2(2n+1)}, \frac{1}{2(2n+1)}, 0\Big\} \\ &= \frac{1}{2(2n+1)} \to 0 \text{ as } n \to \infty.\end{aligned}$$

This shows that $\mathcal{P}y_n \to 0$ as $n \to \infty$.

Also note that

$$\begin{aligned}\mathcal{S}(\mathcal{R}y_n, \mathcal{R}y_n, 0) = \mathcal{S}\Big(\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\Big) &= \max\Big\{\frac{1}{4(2n+1)}, \frac{1}{4(2n+1)}, 0\Big\} \\ &= \frac{1}{4(2n+1)} \to 0 \text{ as } n \to \infty.\end{aligned}$$

This shows that $\mathcal{R}y_n \to 0$ as $n \to \infty$. Thus there exists a sequence $\{y_n\}$ in X such that $\mathcal{P}y_n \to 0$ and $\mathcal{R}y_n \to 0$ as $n \to \infty$. Hence the pair $(\mathcal{P}, \mathcal{R})$ satisfies $(E.A)$ property.

Definition 2.17. ([21]) Consider the class of functions $\Psi = \{\psi | \psi \colon [0, \infty) \to [0, \infty)\}$, which satisfy the following assertions:

($\mathbf{\Psi_1}$) $t_1 \leq t_2$ implies $\psi(t_1) \leq \psi(t_2)$;

($\mathbf{\Psi_2}$) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$;

($\mathbf{\Psi_3}$) $\sum_{n=1}^{\infty} \psi^n(t)$ is convergent for all $t > 0$.

If conditions ($\mathbf{\Psi_1}$)-($\mathbf{\Psi_2}$) hold, then ψ is called a comparison function and if the comparison function satisfies ($\mathbf{\Psi_3}$), then ψ is called a strong comparison function.

Remark 2.18. ([21]) If $\psi \colon [0, \infty) \to [0, \infty)$ is a comparison function, then $\psi(t) < t$ for all $t > 0$, $\psi(0) = 0$ and ψ is right continuous at 0.

Recently, Tiwari and Tripathi [31] introduced the following notion.

Definition 2.19. ([31]) Let $\mathbb{R}_+$ be the set of all non-negative real numbers and A_φ be the collection of all functions $\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the conditions:

(1) α is continuous on $\mathbb{R}_+^4$ (with respect to the Euclidean metric on $\mathbb{R}_+^4$);

(2) for all $u, v \in \mathbb{R}_+$, if

(2_a) $u \leq \alpha(u, v, v, v)$ or

(2_b) $u \leq \alpha(v, u, v, v)$ or

(2_c) $u \leq \alpha(v, v, u, v)$, then $u \leq \varphi(v)$, where φ is a strong comparison function.

If $\varphi(t) = kt$ for $k \in [0, 1)$ and for all $t > 0$, then we have $\alpha \in A_\varphi$.

Now, we define the following implicit relation.

Implicit Relation.

Definition 2.20. Let $\mathbb{R}_+$ be the set of all non-negative real numbers and A_ψ be the collection of all functions $\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the following conditions:

(1) α is continuous on $\mathbb{R}_+^5$ (with respect to the Euclidean metric on $\mathbb{R}_+^5$);

(2) for all $x, y \in \mathbb{R}_+$, if
(α_1) $x \le \alpha(y, x, x, x, x)$ or
(α_2) $x \le \alpha(y, x, y, y, x)$ or
(α_3) $x \le \alpha(x, x, y, x, x)$, then $x \le \psi(y)$, where ψ is a strong comparison function. If $\psi(t) = ht$ for $h \in [0, 1)$ and for all $t > 0$, then we have $\alpha \in A_\psi$.

Lemma 2.21. *([27], Lemma 2.5) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then, we have $\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u)$ for all $u, v \in X$.*

Lemma 2.22. *([27], Lemma 2.12) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $u_n \to u$ and $v_n \to v$ as $n \to \infty$ then $\mathcal{S}(u_n, u_n, v_n) \to \mathcal{S}(u, u, v)$ as $n \to \infty$.*

Lemma 2.23. *([7], Lemma 8) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and A be a nonempty subset of X. Then A is $\mathcal{S}$-closed if and only if for any sequence $\{u_n\}$ in A such that $u_n \to u$ as $n \to \infty$, then $u \in A$.*

Lemma 2.24. *([27]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $r > 0$ and $u \in X$, then the ball $B_{\mathcal{S}}(u, r)$ is an open subset of X.*

Lemma 2.25. *([28]) The limit of a convergent sequence in an $\mathcal{S}$-metric space $(X, \mathcal{S})$ is unique.*

Lemma 2.26. *([27]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then any convergent sequence in X is Cauchy.*

3. Main results

In this section, we shall prove some common fixed point theorems in the setting of S-metric spaces using $(E.A)$-property under an implicit relation.

Theorem 3.1. *Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and let $f, g, P, Q \colon X \to X$ be four self-mappings of X. If there exists some $\alpha \in A_\psi$ such that for all $x, y, z \in X$ satisfying the following conditions:*
(i)

$$\begin{aligned} \mathcal{S}(fx, fy, gz) \;\le\; \alpha\Big(&\mathcal{S}(Px, Py, Qz), \mathcal{S}(gz, gz, fx), \mathcal{S}(gz, gz, Qz), \\ &\mathcal{S}(gz, gz, Px), \\ &\frac{\mathcal{S}(gz, gz, fy)[1 + \mathcal{S}(gz, gz, Qz)]}{[1 + \mathcal{S}(fx, fx, gz)]}\Big), \end{aligned} \tag{3.1}$$

(ii) the pairs (f, P) and (g, Q) are weakly compatible;
(iii) one of the pairs (f, P) and (g, Q) satisfies the $(E.A)$ property;
(iv) $f(X) \subseteq Q(X)$ and $g(X) \subseteq P(X)$.
If the range of one of the mappings P or Q is a complete subspace of $(X, \mathcal{S})$, then f, g, P and Q have a unique common fixed point in X.

Proof. Suppose that the pair (f, P) satisfies the $(E.A)$ property. Then by Definition 2.15, there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} Py_n = u$ for some $u \in X$. Further, since $f(X) \subseteq Q(X)$, there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} Qz_n$. Hence $\lim_{n\to\infty} Qz_n = u$. We claim that

$\lim_{n\to\infty} gz_n = u$. If not, then putting $x = y = y_n$ and $z = z_n$ in inequality (3.1) and using Lemma 2.21 and ($S1$), we have

$$\begin{aligned}
\mathcal{S}(fy_n, fy_n, gz_n) &\leq \alpha\Big(\mathcal{S}(Py_n, Py_n, Qz_n), \mathcal{S}(gz_n, gz_n, fy_n), \mathcal{S}(gz_n, gz_n, Qz_n),\\
&\qquad \mathcal{S}(gz_n, gz_n, Py_n),\\
&\qquad \frac{\mathcal{S}(gz_n, gz_n, fy_n)[1 + \mathcal{S}(gz_n, gz_n, Qz_n)]}{[1 + \mathcal{S}(fy_n, fy_n, gz_n)]}\Big)\\
&= \alpha\Big(\mathcal{S}(fy_n, fy_n, fy_n), \mathcal{S}(gz_n, gz_n, fy_n), \mathcal{S}(gz_n, gz_n, fy_n),\\
&\qquad \mathcal{S}(gz_n, gz_n, fy_n),\\
&\qquad \frac{\mathcal{S}(gz_n, gz_n, fy_n)[1 + \mathcal{S}(gz_n, gz_n, fy_n)]}{[1 + \mathcal{S}(fy_n, fy_n, gz_n)]}\Big)\\
&= \alpha\Big(0, \mathcal{S}(fy_n, fy_n, gz_n), \mathcal{S}(fy_n, fy_n, gz_n), \mathcal{S}(fy_n, fy_n, gz_n),\\
&\qquad \mathcal{S}(fy_n, fy_n, gz_n)\Big). \qquad (3.2)
\end{aligned}$$

Letting $n \to \infty$ in equation (3.2), we obtain

$$\begin{aligned}
\mathcal{S}(u, u, gz_n) &\leq \alpha\Big(0, \mathcal{S}(u, u, gz_n), \mathcal{S}(u, u, gz_n), \mathcal{S}(u, u, gz_n)\\
&\qquad \mathcal{S}(u, u, gz_n)\Big), \qquad (3.3)
\end{aligned}$$

by Definition of (α_1), we get $\mathcal{S}(u, u, gz_n) \leq \psi(0) = 0$, which implies that $\mathcal{S}(u, u, gz_n) = 0$. Thus $\lim_{n\to\infty} gz_n = u$ and hence $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gz_n = u$.

Now, first we assume that $Q(X)$ is a complete subspace of $(X, \mathcal{S})$, then $u = Qv$ for some $v \in X$. Subsequently, we have

$$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} Py_n = \lim_{n\to\infty} Qz_n = Qv = u.$$

We claim that $gu = Qu$. For this, putting $x = y = y_n$ and $z = v$ in inequality (3.1) and using Lemma 2.21, we have

$$\begin{aligned}
\mathcal{S}(fy_n, fy_n, gv) &\leq \alpha\Big(\mathcal{S}(Py_n, Py_n, Qv), \mathcal{S}(gv, gv, fy_n), \mathcal{S}(gv, gv, Qv),\\
&\qquad \mathcal{S}(gv, gv, Py_n),\\
&\qquad \frac{\mathcal{S}(gv, gv, fy_n)[1 + \mathcal{S}(gv, gv, Qv)]}{[1 + \mathcal{S}(fy_n, fy_n, gv)]}\Big)\\
&= \alpha\Big(\mathcal{S}(fy_n, fy_n, Qv), \mathcal{S}(gv, gv, fy_n), \mathcal{S}(gv, gv, Qv),\\
&\qquad \mathcal{S}(gv, gv, Py_n),\\
&\qquad \frac{\mathcal{S}(gv, gv, fy_n)[1 + \mathcal{S}(gv, gv, Qv)]}{[1 + \mathcal{S}(fy_n, fy_n, gv)]}\Big). \qquad (3.4)
\end{aligned}$$

Letting $n \to \infty$ in (3.4) and using Lemma 2.21 and $(S1)$, we get

$$\begin{aligned}\mathcal{S}(Qv, Qv, gv) &\le \alpha\Big(\mathcal{S}(Qv, Qv, Qv), \mathcal{S}(gv, gv, Qv), \mathcal{S}(gv, gv, Qv),\\ &\quad \mathcal{S}(gv, gv, Qv),\\ &\quad \frac{\mathcal{S}(gv, gv, Qv)[1 + \mathcal{S}(gv, gv, Qv)]}{[1 + \mathcal{S}(Qv, Qv, gv)]}\Big)\\ &= \alpha\Big(0, \mathcal{S}(Qv, Qv, gv), \mathcal{S}(Qv, Qv, gv), (Qv, Qv, gv),\\ &\quad \mathcal{S}(Qv, Qv, gv)\Big), \qquad (3.5)\end{aligned}$$

by Definition of (α_1), we get $\mathcal{S}(Qv, Qv, gv) \le \psi(0) = 0$, which implies that $\mathcal{S}(Qv, Qv, gv) = 0$, that is, $Qv = gv$. Thus $Qv = gv = u$. Hence v is a coincidence point of the mappings g and Q, that is, the pair (g, Q). Since the pair (g, Q) is weak compatible, so by weak compatibility of this pair, we have $gQv = Qgv$ or $gu = Qu$.

On the other hand, since $g(X) \subseteq P(X)$, there exists $w \in X$ such that $gv = Pw$. Thus $Qv = gv = Pw = u$. Let us show that w is a coincidence point of the pair (f, P), that is, $fw = Pw = u$. If not, then putting $x = y = w$ and $z = v$ in inequality (3.1) and using Lemma 2.21 and $(S1)$, we get

$$\begin{aligned}\mathcal{S}(fw, fw, gv) &\le \alpha\Big(\mathcal{S}(Pw, Pw, Qv), \mathcal{S}(gv, gv, fw), \mathcal{S}(gv, gv, Qv),\\ &\quad \mathcal{S}(gv, gv, Pw), \frac{\mathcal{S}(gv, gv, fw)[1 + \mathcal{S}(gv, gv, Qv)]}{[1 + \mathcal{S}(fw, fw, gv)]}\Big)\\ &= \alpha\Big(\mathcal{S}(u, u, u), \mathcal{S}(u, u, fw), \mathcal{S}(u, u, u), \mathcal{S}(u, u, u),\\ &\quad \frac{\mathcal{S}(u, u, fw)[1 + \mathcal{S}(u, u, u)]}{[1 + \mathcal{S}(fw, fw, u)]}\Big)\\ &= \alpha\Big(0, \mathcal{S}(fw, fw, u), 0, 0, \frac{\mathcal{S}(fw, fw, u)}{[1 + \mathcal{S}(fw, fw, u)]}\Big)\\ &\le \alpha\Big(0, \mathcal{S}(fw, fw, u), 0, 0, \mathcal{S}(fw, fw, u)\Big),\end{aligned}$$

or

$$\mathcal{S}(fw, fw, u) \le \alpha\Big(0, \mathcal{S}(fw, fw, u), 0, 0, \mathcal{S}(fw, fw, u)\Big), \qquad (3.6)$$

by Definition of (α_2), we get $\mathcal{S}(fw, fw, u) \le \psi(0) = 0$, which implies that $\mathcal{S}(fw, fw, u) = 0$, that is, $fw = u$. Thus $fw = Pw = u$. Thus w is a coincidence point of the mappings f and P. Further, the weak compatibility of the pair (f, P) implies that $fPw = Pfw$ or $fu = Pu$. Hence u is a common coincidence point of f, g, P and Q.

Now to show that u is a common fixed point of f, g, P and Q. For this, we put $x = y = w$ and $z = u$ in (3.1) and using Lemma 2.21 and ($S1$), we get

$$\begin{aligned}
\mathcal{S}(fw, fw, gu) &\leq \alpha\Big(\mathcal{S}(Pw, Pw, Qu), \mathcal{S}(gu, gu, fw), \mathcal{S}(gu, gu, Qu), \\
&\qquad \mathcal{S}(gu, gu, Pw), \frac{\mathcal{S}(gu, gu, fw)[1 + \mathcal{S}(gu, gu, Qu)]}{[1 + \mathcal{S}(fw, fw, gu)]}\Big) \\
&= \alpha\Big(\mathcal{S}(u, u, gu), \mathcal{S}(gu, gu, u), \mathcal{S}(gu, gu, gu), \mathcal{S}(gu, gu, u), \\
&\qquad \frac{\mathcal{S}(gu, gu, u)[1 + \mathcal{S}(gu, gu, gu)]}{[1 + \mathcal{S}(u, u, gu)]}\Big) \\
&= \alpha\Big(\mathcal{S}(u, u, gu), \mathcal{S}(u, u, gu), 0, \mathcal{S}(u, u, gu), \frac{\mathcal{S}(u, u, gu)}{[1 + \mathcal{S}(u, u, gu)]}\Big) \\
&\leq \alpha\Big(\mathcal{S}(u, u, gu), \mathcal{S}(u, u, gu), 0, \mathcal{S}(u, u, gu), \mathcal{S}(u, u, gu)\Big),
\end{aligned}$$

or

$$\mathcal{S}(u, u, gu) \leq \alpha\Big(\mathcal{S}(u, u, gu), \mathcal{S}(u, u, gu), 0, \mathcal{S}(u, u, gu), \mathcal{S}(u, u, gu)\Big), \tag{3.7}$$

by Definition of (α_3), we get $\mathcal{S}(u, u, gu) \leq \psi(0) = 0$, which implies that $\mathcal{S}(u, u, gu) = 0$, that is, $gu = u$. Consequently, $fu = gu = Pu = Qu = u$. This shows that u is a common fixed point of the mappings f, g, P and Q.

Similar argument arises if we assume that $P(X)$ is a complete subspace of $(X, \mathcal{S})$.

Similarly, the property $(E.A)$ of the pair (g, Q) will give the similar result.

Now, we show the uniqueness of the common fixed point. For this, let us suppose that u' be another common fixed point of f, g, P and Q such that $fu' = gu' = Pu' = Qu' = u'$ with $u' \neq u$. By inequality (3.1) and using Lemma 2.21 and ($S1$) for $x = y = u'$ and $z = u$, we get

$$\begin{aligned}
\mathcal{S}(u', u', u) &= \mathcal{S}(fu', fu', gu) \\
&\leq \alpha\Big(\mathcal{S}(Pu', Pu', Qu), \mathcal{S}(gu, gu, fu'), \mathcal{S}(gu, gu, Qu), \\
&\qquad \mathcal{S}(gu, gu, Pu'), \\
&\qquad \frac{\mathcal{S}(gu, gu, fu')[1 + \mathcal{S}(gu, gu, Qu)]}{[1 + \mathcal{S}(fu', fu', gu)]}\Big) \\
&= \alpha\Big(\mathcal{S}(u', u', u), \mathcal{S}(u, u, u'), \mathcal{S}(u, u, u), \mathcal{S}(u, u, u'), \\
&\qquad \frac{\mathcal{S}(u, u, u')[1 + \mathcal{S}(u, u, u)]}{[1 + \mathcal{S}(u', u', u)]}\Big) \\
&= \alpha\Big(\mathcal{S}(u', u', u), \mathcal{S}(u', u', u), 0, \mathcal{S}(u', u', u), \\
&\qquad \frac{\mathcal{S}(u', u', u)}{[1 + \mathcal{S}(u', u', u)]}\Big) \\
&\leq \alpha\Big(\mathcal{S}(u', u', u), \mathcal{S}(u', u', u), 0, \mathcal{S}(u', u', u), \mathcal{S}(u', u', u)\Big),
\end{aligned}$$

by Definition of (α_3), we get $\mathcal{S}(u', u', u) \leq \psi(0) = 0$, which implies that $\mathcal{S}(u', u', u) = 0$, that is, $u' = u$. This shows that the common fixed point of the mappings f, g, P and Q is unique. This completes the proof. $\square$

If we take $Q = P$ and $g = f$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2. *Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and let $f, P \colon X \to X$ be two self-mappings of X. If there exists some $\alpha \in A_\psi$ such that for all $x, y, z \in X$ satisfying the following conditions:*

(i)

$$\begin{aligned}\mathcal{S}(fx, fy, fz) \;\leq\; \alpha\Big(&\mathcal{S}(Px, Py, Pz), \mathcal{S}(fz, fz, fx), \mathcal{S}(fz, fz, Pz),\\ &\mathcal{S}(fz, fz, Px),\\ &\frac{\mathcal{S}(fz, fz, fy)[1 + \mathcal{S}(fz, fz, Pz)]}{[1 + \mathcal{S}(fx, fx, fz)]}\Big),\end{aligned}$$

(ii) *the pair (f, P) is weakly compatible;*

(iii) *the pair (f, P) satisfies $(E.A)$ property;*

(iv) $f(X) \subseteq P(X)$.

If the range of P is a complete subspace of $(X, \mathcal{S})$, then f and P have a unique common fixed point in X.

If we take $g = f$ and $\alpha(t_1, t_2, t_3, t_4, t_5) = \alpha(t_1, t_2, t_3, t_4)$ in Theorem 3.1, then we obtain the following result.

Corollary 3.3. *Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and let $f, P, Q \colon X \to X$ be three self-mappings of X. If there exists some $\alpha \in A_\psi$ such that for all $x, y, z \in X$ satisfying the following conditions:*

(i)

$$\begin{aligned}\mathcal{S}(fx, fy, fz) \;\leq\; \alpha\Big(&\mathcal{S}(Px, Py, Qz), \mathcal{S}(fz, fz, fx), \mathcal{S}(fz, fz, Qz),\\ &\mathcal{S}(fz, fz, Px)\Big),\end{aligned}$$

(ii) *the pairs (f, P) and (f, Q) are weakly compatible;*

(iii) *one of the pairs (f, P) and (f, Q) satisfies the $(E.A)$ property;*

(iv) $f(X) \subseteq P(X)$ *and* $f(X) \subseteq Q(X)$.

If the range of one of the mappings P or Q is a complete subspace of $(X, \mathcal{S})$, then f, P and Q have a unique common fixed point in X.

If we take $g = f$ and $\alpha(t_1, t_2, t_3, t_4, t_5) = \alpha(t_1, t_2, t_3)$ in Theorem 3.1, then we obtain the following result.

Corollary 3.4. *Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and let $f, P, Q \colon X \to X$ be three self-mappings of X. If there exists some $\alpha \in A_\psi$ such that for all $x, y, z \in X$ satisfying the following conditions:*

(i)

$$\mathcal{S}(fx, fy, fz) \leq \alpha\Big(\mathcal{S}(Px, Py, Qz), \mathcal{S}(fz, fz, fx), \mathcal{S}(fz, fz, Qz)\Big),$$

(ii) the pairs (f, P) and (f, Q) are weakly compatible;
(iii) one of the pairs (f, P) and (f, Q) satisfies the $(E.A)$ property;
(iv) $f(X) \subseteq P(X)$ and $f(X) \subseteq Q(X)$.
If the range of one of the mappings P or Q is a complete subspace of $(X, \mathcal{S})$, then f, P and Q have a unique common fixed point in X.

Remark 3.5. By using similar fashion we can find some more results from Theorem 3.1.

If we take $P = I$ (where I is an identity mapping) and $\alpha(t_1, t_2, t_3, t_4, t_5) = k\, t_1$, where $k \in [0, 1)$ in Corollary 3.2, then we obtain the following result.

Corollary 3.6. *Let $(X, \mathcal{S})$ be a complete $\mathcal{S}$-metric space and let $f\colon X \to X$ be a self-mapping of X satisfying the following condition:*

$$\mathcal{S}(fx, fy, fz) \leq k\, \mathcal{S}(x, y, z),$$

for all $x, y, z \in X$, where $k \in [0, 1)$ is a constant. Then f has a unique fixed point in X.

Remark 3.7. Corollary 3.6 extends the well-known Banach fixed point theorem [4] from complete metric space to the setting of complete $\mathcal{S}$-metric space.

Corollary 3.8. *Let $(X, \mathcal{S})$ be a complete $\mathcal{S}$-metric space and let $f\colon X \to X$ be a self-mapping of X satisfying the contractive condition:*

$$\mathcal{S}(f^n x, f^n y, f^n z) \leq k\, \mathcal{S}(x, y, z),$$

for all $x, y, z \in X$, where n is some positive integer and $k \in [0, 1)$ is a constant. Then f has a unique fixed point in X.

Proof. By Corollary 3.6, there exists $z \in X$ such that $f^n z = z$. Then

$$\begin{aligned} \mathcal{S}(fz, fz, z) &= S(ff^n z, ff^n z, f^n z) \\ &= \mathcal{S}(f^n fz, f^n fz, f^n z) \\ &\leq k\, \mathcal{S}(fz, fz, z), \end{aligned}$$

which is a contradiction since $0 \leq k < 1$ and so $S(fz, fz, z) = 0$, that is, $fz = z$. This shows that f has a unique fixed point in X. This completes the proof. □

Remark 3.9. Corollary 3.6 is a special case of Corollary 3.8 for $n = 1$.

Now, we give an example to establish the validity of Theorem 3.1 and Corollary 3.6.

Example 3.10. Let $X = [3, 15]$. We define the function $\mathcal{S}\colon X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0, & \text{if } \ x = y = z, \\ \max\{x, y, z\}, & \text{if } \ \text{otherwise}, \end{cases}$$

for all $x, y, z \in X$, then $\mathcal{S}$ is an $\mathcal{S}$-metric on X. Define four self-maps $f, g, P, Q\colon X \to X$ on X by

$$f(x) = \begin{cases} 3, & \text{if } \ x \in \{3\} \cup (5, 15], \\ 4, & \text{if } x \in (3, 5], \end{cases}$$

$$g(x) = \begin{cases} 3, & \text{if } x \in \{3\} \cup (5, 15], \\ 5, & \text{if } x \in (3, 5], \end{cases}$$

$$P(x) = \begin{cases} 3, & \text{if } x = 3, \\ 10, & \text{if } x \in (3, 5], \\ \frac{x+1}{2}, & \text{if } x \in (5, 15], \end{cases}$$

$$Q(x) = \begin{cases} 3, & \text{if } x = 3, \\ 5, & \text{if } x \in (3, 5], \\ \frac{x+1}{2}, & \text{if } x \in (5, 15], \end{cases}$$

(1) Now, we take the sequences $\{x_n\} = \{4 + \frac{1}{n}\}$ and $\{y_n\} = \{5\}$. Now, we observe that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Px_n = 3 \in X.$$

Thus the pair (f, P) satisfies $(E.A)$ property.

Similarly, we have

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Qy_n = 5 \in X.$$

Thus the pair (g, Q) also satisfies $(E.A)$ property.

Again, we observe that the pair of mappings (f, P) commute at 3 which is the coincidence point of the mappings f and P and the pair of mappings (g, Q) commute at 5 which is the coincidence point of the mappings g and Q.

Also,

$$f(X) = \{3, 4\} \subseteq [3, 8] \cup (8, 10] = Q(X) \text{ and } g(X) = \{3, 5\} \subseteq [3, 8] \cup \{10\} = P(X).$$

Now, we can verify the contractive condition (3.1) of Theorem 3.1 for the case $x, y, z \in [3, 5]$, by a simple calculation we see that

$$\mathcal{S}(fx, fy, gz) = 5,\ \mathcal{S}(Px, Py, Qz) = 10,\ \mathcal{S}(gz, gz, fx) = 5,\ \mathcal{S}(gz, gz, Qz) = 10,$$

$$\mathcal{S}(gz, gz, Px) = 10,\ \mathcal{S}(gz, gz, fy) = 5,\ \mathcal{S}(fx, fx, gz) = 5,$$

$$\frac{\mathcal{S}(gz, gz, fy)[1 + \mathcal{S}(gz, gz, Qz)]}{[1 + \mathcal{S}(fx, fx, gz)]} = \frac{5[1 + 10]}{[1 + 5]} = \frac{55}{6}.$$

Now using the inequality (3.1), which yields

$$5 \leq \alpha\Big(10, 5, 10, 10, \frac{55}{6}\Big),$$

where $\alpha(x, y, z, t, w) = \max\{x, y, z, t, w\}$ and $\psi(t) = \frac{2t}{3}$. Thus, we see that

$$5 \leq \psi(10) = \frac{20}{3} \text{ or } 15 \leq 20,$$

which is true. Similarly, we can verify for other cases. Thus all the conditions of Theorem 3.1 are satisfied and 3 is the unique common fixed point of the mappings f, g, P and Q.

(2) Now using inequality of Corollary 3.6, if we take $x = 6$, $y = 7$ and $z = 4$, then we see that $f(6) = 3$, $f(7) = 3$ and $g(4) = 5$. Now, we have

$$\mathcal{S}(fx, fy, gz) = \max\{3, 3, 5\} = 5 \quad \text{and} \quad \mathcal{S}(x, y, z) = \max\{6, 7, 4\} = 7.$$

Consequently, we have

$$\mathcal{S}(fx, fy, gz) = 5 \leq k\,\mathcal{S}(x, y, z) = 7\,k,$$

or,

$$5 \leq 7k,$$

or,

$$k \geq \frac{5}{7}.$$

If we take $0 \leq k < 1$, then all the conditions of Corollary 3.6 are satisfied and 3 is the unique fixed point of f. Hence we conclude that

$$\mathcal{S}(fx, fy, gz) \leq k\,\mathcal{S}(x, y, z).$$

4. Conclusion

In this paper, we prove some common fixed point theorems in the setting of S-metric spaces using $(E.A)$-property via an implicit relation. We give an example to demonstrate the validity of the main result and also give some corollaries of the established result. The results presented in this paper extend, generalize and enrich several results from the existing literature.

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