Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 9 (2022) 91-100

SEMIDIRECT PRODUCT OF WEAK INVERSE PROPERTY POWER ASSOCCIATIVE CONJUGACY CLOSED (WIP PACC) LOOPS

OLUFEMI GEORGE

ABSTRACT. In this work, it is shown that if A, K and Q are loops such that $Q = A \times_\theta K$, where $\theta : A \to \mathrm{Sym}K$. Then Q is weak inverse property power associative conjugacy closed loop if and only if A and K are weak inverse property power associative conjugacy closed loops, $\theta(x)$ is a nuclear automorphism of K in the sense that $k^{-1}k\theta(x) \in N(K)$ for each $x \in A,\ k \in K$ and $\theta(xy) = \theta(x)\theta(y)$.

1. Introduction and preliminaries

Phillips in [10] axiomatized the variety of weak inverse property power associative conjugacy closed loops by the LWPC; $(xy \cdot x) \cdot xz = x((yx \cdot x)z)$ and the RWPC; $zx \cdot (x \cdot yx) = (z(x \cdot xy))x$ identities. George and Jaiyeola in [4] using a generalized and modified nuclear identification model originally introduced by Drápal and Jedlička [2] discovered twelve new loop identities which they christened loops of second Bol-Moufang type. A loop is said to be of second Bol-Moufang type if it contains three variables in the same order on both sides of the equation and with one of the variables appearing three times.
Coincidentally, the LWPC and RWPC identities are among the twelve newly nuclearly identified loops.
The main purpose of this paper is to present a necessary and sufficient conditions for the semi-direct product of loops to produce WIP PACC loop.

A loop Q is a set with a binary operation $\cdot$ such that for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions x and y in Q and there exists $e \in Q$ such that $ex = xe = x$ for any $x \in Q$. The unique solutions are given by $x = a \setminus b$ and $y = b/a$. The reader can consult the books [1, 9] for a general background in loop theory. In a loop Q with identity 1, for all $x \in Q$ there exists a unique right inverse element x^ρ and a unique left inverse element x^λ such that $xx^\rho = x^\lambda x = 1$.

Date: Received: Oct 6, 2022; Accepted: Oct 30, 2022.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 20N02, 20N05.
Key words and phrases. left (right) conjugacy closed loop, power associative, weak inverse property, LWPC-loop, RWPC-loop, Semidirect Product.

Note that x^ρ is not always equal to x^λ in a loop, when they are equal we write $x^\rho = x^\lambda = x^{-1}$ and say the loop has two sided inverse.

The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop Q are defined by

$$N_\lambda(Q) = \{a \in Q : \ a \cdot xy = ax \cdot y \ \ \forall\ x,\ y \in Q\},$$
$$N_\mu(Q) = \{a \in Q : \ xa \cdot y = x \cdot ay \ \ \forall\ x,\ y \in Q\},$$
$$N_\rho(Q) = \{a \in Q : \ xy \cdot a = x \cdot ya \ \ \forall\ x,\ y \in Q\}.$$

The intersection

$$N(Q) = N_\rho(Q) \cap N_\lambda(Q) \cap N_\mu(Q)$$

is called the nucleus of Q.

A triple of bijections $(U,\ V,\ W)$ is called an autotopism of a loop Q provided that

$$xU \cdot yV = (xy)W. \tag{1.1}$$

for all $x,\ y \in Q$.

It is easy to see that

$$a \in N_\lambda(Q) \Leftrightarrow (L(a),\ I,\ L(a)) \in Atp(Q) \tag{1.2}$$
$$a \in N_\mu(Q) \Leftrightarrow (R^{-1}(a),\ L(a),\ I) \in Atp(Q) \tag{1.3}$$
$$a \in N_\rho(Q) \Leftrightarrow (I,\ R(a),\ R(a)) \in Atp(Q) \tag{1.4}$$

Conjugacy closed loops (CC-loop) are loops satisfying the following two identities:

$$zy \cdot x = zx \ \cdot x\backslash(yx). \qquad (RCC)$$
$$x \cdot yz = (xy)/x \ \cdot xz. \qquad (LCC)$$

A loop $(Q, \cdot)$ is called an extra loop if it satisfies any of the following for all $x, y, z \in Q$:

$$xy \cdot xz = x(yx \cdot z). \tag{1.5}$$
$$yx \cdot zx = (y \cdot xz)x. \tag{1.6}$$

A loop is a weak inverse property loop if it satisfies any one of the equivalent identities:

$$x(yx)^\rho = y^\rho \quad or \quad (xy)^\lambda x = y^\lambda. \tag{1.7}$$

A loop $(Q,\ \cdot)$ is power associative if subloops generated by every single elements are groups. This is easily seen to be equivalent to $x^{m+n} = x^m \cdot x^n$ for every $x \in Q$ and for all $m,\ n \in Z$.

By [10], a loop Q is a WIP PACC-loop if and only if it fulfils the laws

$$(xy \cdot x) \ \cdot xz = x((yx \cdot x)z) \text{ and} \qquad \text{(LWPC)}$$
$$zx \ \cdot (x \cdot yx) = (z(x \cdot xy))x \qquad \text{(RWPC)}$$

Definition 1.1. [6]. Let A and K be loops and let $\theta : A \to \mathrm{Sym}K$ such that $(e_K)\theta(a) = e_K$ and $k\theta(e_A) = k$. The external semi-direct product $A \times_\theta K$ is the set $A \times K$, with the binary operation

$$(a_1,\ k_1) \cdot (a_2,\ k_2) = (a_1a_2,\ \ k_1\theta(a_2) \cdot k_2) \tag{1.8}$$

for all $a_1,\ a_2 \in A,\ k_1,\ k_2 \in K$.

It is easy to see that $A \times_\theta K$ is a loop with identity element $(e_A,\ e_K)$, right and left division operations given by:

$$(a_1,\ k_1)/(a_2,\ k_2) = (a_1/a_2,\ (k_1/k_2)\theta_{a_2}^{-1}) \tag{1.9}$$

$$(a_1,\ k_1)\backslash(a_2,\ k_2) = (a_1\backslash a_2,\ [(k_1)\theta_{a_1\backslash a_2}]\backslash k_2). \tag{1.10}$$

Theorem 1.2. [5]. *If $Q = A \times K$ is the external semidirect product of loops A and K, then Q is isomorphic to the internal semidirect product of subloops $A \times \{1\}$ and $\{1\} \times K$.*

2. Main results

Proposition 2.1. *Let Q be an LWPC (RWPC) loop. Then $x^\lambda = x^\rho = x^{-1}$.*

Proof. Put $z = 1$ and $y = x^\lambda$ in LWPC (RWPC) identity. □

The following lemma gives an autotopism characterization of a WIP PACC loop.

Lemma 2.2. [3] *Let $(Q,\ \cdot)$ be a loop.*
(a) $(Q,\ \cdot)$ is an LWPC loop $\Leftrightarrow A(x) = (R^{-2}(x)L(x)R(x),\ L(x),\ L(x)) \in\ AtpQ$
(b) $(Q,\ \cdot)$ is an RWPC loop $\Leftrightarrow B(x) = (R(x),\ L^{-2}(x)R(x)L(x),\ R(x)) \in\ AtpQ$.

We now determine when the semi-direct products of loops produce WIP PACC loop.

Lemma 2.3. *Let Q, A, and K be loops such that $Q = A \times_\theta K$, where $\theta : A \to SymK$. Then Q is an LWPC loop if and only if*
(i) A is an LWPC loop, and
(ii) the equation

$$((k_1\theta(a_2){\cdot}k_2)\theta(a_1){\cdot}k_1)\theta(a_1a_3){\cdot}(k_1\theta(a_3){\cdot}k_3) = k_1\theta((a_2a_1{\cdot}a_1)a_3){\cdot}(((k_2\theta(a_1){\cdot}k_1)\theta(a_1){\cdot}k_1)\theta(a_3){\cdot}k_3) \tag{2.1}$$

holds for all $a_1,\ a_2,\ a_3 \in A$ and $k_1,\ k_2,\ k_3 \in K$.

Proof. Let $x = (a_1,\ k_1),\ y = (a_2,\ k_2)$, and $z = (a_3,\ k_3) \in Q$.
Now use equation (1.8) to exhibit the LWPC identity;

$$(xy{\cdot}x){\cdot}xz = ((a_1a_2{\cdot}a_1){\cdot}a_1a_3,\ ((k_1\theta(a_2){\cdot}k_2)\theta(a_1){\cdot}k_1)\theta(a_1a_3){\cdot}(k_1\theta(a_3){\cdot}k_3). \tag{2.2}$$

$$x((yx{\cdot}x)z) = (a_1((a_2a_1{\cdot}a_1)a_3),\ k_1\theta((a_2a_1{\cdot}a_1)a_3){\cdot}(((k_2\theta(a_1){\cdot}k_1)\theta(a_1){\cdot}k_1)\theta(a_3){\cdot}k_3). \tag{2.3}$$

Comparing the first and second components of equation (2.2) and (2.3), we obtain

$$(a_1a_2 \cdot a_1) \cdot (a_1a_3) = a_1((a_2a_1 \cdot a_1)a_3) \tag{2.4}$$

and equation (2.1). □

Lemma 2.4. *Let Q, A, and K be loops such that Q is the external semi-direct product of A and K. If $\theta(x)$ is an automorphism of K for all $x \in A$, if $\theta(yx) = \theta(y)\theta(x)$ for all $x,\ y \in A$, then*

$$A(k,\ x,\ y) = (R^{-1}(k)R^{-1}(k\theta(x))L(k\theta(y)\theta(x)\theta(x))R(k\theta(x)),\ L(k),\ L(k\theta(y)\theta(x)\theta(x)))$$

for all $x,\ y \in A$ and all $k \in K$.

Proof. The LWPC identity is equivalent to

$$(x((y/x)/x))x \cdot xz = x \cdot yz. \tag{2.5}$$

Let $x = (a_1, k_1),\ y = (a_2, k_2),\ z = (a_3, k_3)$ for $a_1, a_2, a_3 \in A$ and $k_1, k_2, k_3 \in K$. Now using equation (1.9) and (1.8) to express (2.5), we have

$$[(a_2, k_2)/(a_1, k_1)]/(a_1, k_1) = ((a_2/a_1)/a_1,\ (((k_2/k_1)\theta^{-1}(a_1))k_1)\theta^{-1}(a_1)),$$

$$(a_1, k_1)[((a_2, k_2)/(a_1, k_1))/(a_1, k_1)] = (a_1a_4,\ k_1\theta(a_4) \cdot (((k_2/k_1)\theta^{-1}(a_1))/k_1)\theta^{-1}(a_1)),$$

where $a_4 = (a_2/a_1)/a_1$ for $a_4 \in A$.

$$\begin{aligned}&[(a_1, k_1)(((a_2, k_2)/(a_1, k_1))/(a_1, k_1))](a_1, k_1)\\ =&(a_1a_4 \cdot a_1,\ (k_1\theta(a_4)\theta(a_1) \cdot (((k_2/k_1)\theta^{-1}(a_1))/k_1) \cdot k_1),\end{aligned}$$

$$\begin{aligned}&[(a_1, k_1)(((a_2, k_2)/(a_1, k_1))/(a_1, k_1))](a_1, k_1) \cdot\ (a_1, k_1)(a_3, k_3)\\ =&((a_1a_4 \cdot a_1) \cdot a_1a_3, (k_1\theta(a_4)\theta(a_1) \cdot (((k_2/k_1)\theta^{-1}(a_1))/k_1) \cdot k_1)\theta(a_1)(a_3) \cdot (k_1\theta(a_3)) \cdot k_3).\end{aligned} \tag{2.6}$$

$$(a_1, k_1)[(a_2, k_2)(a_3, k_3)] = (a_1 \cdot a_2a_3),\ k_1\theta(a_2a_3) \cdot (k_2\theta(a_3) \cdot k_3)). \tag{2.7}$$

By comparing the second components of (2.6) and (2.7) and recall that $a_4 = (a_2/a_1)/a_1$, then set $a_2 = a_2a_1 \cdot a_1$ and note that $\theta(yx) = \theta(y)\theta(x) \Rightarrow \theta(xy \cdot x) = \theta(x)\theta(y)\theta(x)$, we therefore have

$$[(k_1\theta(a_2)\theta(a_1) \cdot ((k_2/k_1\theta^{-1}(a_1))/k_1)) \cdot k_1]\theta(a_1) \cdot k_1k_3\theta^{-1}(a_3) = k_1\theta(a_2a_1 \cdot a_1) \cdot k_2k_3\theta^{-1}(a_3),$$
$$[(k_1\theta(a_2)\theta(a_1)\theta(a_1) \cdot ((k_2/k_1)/k_1\theta(a_1))]k_1\theta(a_1) \cdot (k_1k_3) = k_1\theta(a_2)\theta(a_1)\theta(a_1)(k_2k_3),$$
$$[(k\theta(y)\theta(x)\theta(x) \cdot ((k_2/k)/k\theta(x))]k\theta(x) \cdot (kk_3) = k\theta(y)\theta(x)\theta(x) \cdot (k_2k_3)$$

and the result follows. □

Theorem 2.5. *Let Q, A, and K be loops such that $Q = A \times_\theta K$, where $\theta : A \to SymK$. Then Q is an LWPC loop if and only if*
(i) A is an LWPC loop,

(ii) $\theta(xy) = \theta(x)\theta(y)$,

(iii) $A(k,\ x,\ y) = (R^{-1}(k)R^{-1}(k\theta(x))L(k\theta(y)\theta(x)\theta(x))R(k\theta(x)),\ L(k),\ L(k\theta(y)\theta(x)\theta(x)))$ is an autotopism of K for all $x,\ y \in A$ and all $k \in K$.

Proof. (i) and (iii) is from Lemmas 2.3 and 2.4, we only need to show that (ii) holds whenever Q is an LWPC loop.
Now set $k_1 = k_3 = e_K$ in (2.1) and replacing $k_2(a_1)$ by k gives

$$\theta(a_1a_3) = \theta(a_1)\theta(a_3).$$

□

Example 2.6. The example in Table 1 resp.(Table 2) is an LWPC (RWPC) loop of order 8.

Using the Groups, Algorithms, Programming (GAP Package) [8] and Library of GAP-LOOPS Package [7], $Z(Q) = N(Q) = \{1, 8\}$.

.	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	6	7	4	5	3
3	3	8	1	7	6	5	4	2
4	4	7	6	8	1	2	3	5
5	5	6	7	1	8	3	2	4
6	6	5	4	2	3	8	1	7
7	7	4	5	3	2	1	8	6
8	8	3	2	5	4	7	6	1

TABLE 1. LWPC-loop Q of order 8

.	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	8	1	6	7	4	5	2
4	4	7	6	1	8	2	3	5
5	5	6	7	8	1	3	2	4
6	6	5	4	3	2	8	1	7
7	7	4	5	2	3	1	8	6
8	8	3	2	5	4	7	6	1

TABLE 2. RWPC-loop Q of order 8

Lemma 2.7. *Let Q, A, and K be loops such that $Q = A \times_\theta K$, where $\theta : A \to SymK$. Then Q is an RWPC loop if and only if*
(i) A is an RWPC loop, and
(ii) the equation

$$\begin{aligned}&[k_3\theta(a_1)\cdot k_1)\theta(a_1\cdot a_2a_1)\cdot((k_1\theta(a_2a_1)\cdot k_2\theta(a_1)\cdot k_1)]\\ =&(k_3\theta(a_1\cdot a_1a_2)\cdot[k_1\theta(a_1a_2)\cdot(k_1\theta(a_2)\cdot k_2))]\theta(a_1)\cdot k_1\end{aligned} \tag{2.8}$$

holds for all $a_1,\ a_2,\ a_3 \in A$ and $k_1,\ k_2,\ k_3 \in K$.

Proof. Let $x = (a_1,\ k_1)$, $y = (a_2,\ k_2)$, and $z = (a_3,\ k_3) \in Q$.
Now use equation (1.8) to exhibit the RWPC identity, and comparing the first and the second components yields

$$a_3a_1\cdot(a_1\cdot a_2a_1) = (a_3(a_1\cdot a_1a_2))a_1 \tag{2.9}$$

and equation (2.8). □

Lemma 2.8. *Let Q, A, and K be loops such that Q is the external semi-direct product of A and K. If $\theta(x)$ is an automorphism of K for all $x \in A$ and $\theta(yx) = \theta(y)\theta(x)$ for all $x,\ y \in A$, then*

$$B(k,\ x,\ y) = (R(k),\ L^{-1}(k)L^{-1}(k\theta(x))R(k\theta(x)\theta(y)\theta(x))L(k\theta(x)),\ R(k\theta(x)\theta(y)\theta(x)))$$

is an autotopisms of K for all $x,\ y \in A$ and all $k \in K$.

Proof. Since $Q = A \times_\theta K$, then by Lemma 2.7, equation (2.8) holds for all $a_1,\ a_2,\ a_3 \in A$, all $k_1,\ k_2,\ k_3 \in K$.

Equation (2.8) holds if and only if

$$\begin{aligned}&(k_3\theta(a_1)\theta(a_1)\theta(a_2)\theta(a_1)\cdot k_1\theta(a_1)\theta(a_2)\theta(a_1))\cdot(k_1\theta(a_2)\theta(a_1)\cdot(k_2\theta(a_1)\cdot k_1))\\ =&(k_3\theta(a_1)\theta(a_1)\theta(a_2)\theta(a_1)\cdot(k_1\theta(a_1)\theta(a_2)\theta(a_1)\cdot(k_1\theta(a_2)\theta(a_1)\cdot k_2\theta(a_1))))\cdot k_1.\end{aligned}$$

$$\begin{aligned}&(k_3\theta(a_1)\theta(a_1)\theta(a_2)\cdot k_1\theta(a_1)\theta(a_2))\cdot(k_1\theta(a_2)\cdot(k_2k_1\theta^{-1}(a_1)))\\ =&(k_3\theta(a_1)\theta(a_1)\theta(a_2)\cdot(k_1\theta(a_1)\theta(a_2)(k_1\theta(a_2))\cdot k_2)\cdot k_1\theta^{-1}(a_1).\end{aligned} \tag{2.10}$$

Put $k_1 = k_1\theta(a_1)$ and set $w = k_3\theta(a_1)\theta(a_1)\theta(a_2), u = k_1\theta(a_1)\theta(a_1)\theta(a_2),\ v = k_2$ in equation (2.10)) to get

$$(wu)(u\theta^{-1}(a) \cdot vu\theta^{-1}(b)\theta^{-1}(a)\theta^{-1}(a)) = (w(u \cdot (u\theta^{-1}(a)v)u\theta^{-1}(b)\theta^{-1}(a)\theta^{-1}(a) \tag{2.11}$$

for all $a,\ b \in A$ and all $u,\ v,\ w \in K$.
From the hypothesis, we have

$$\theta(yx) = \theta(y)\theta(x) \Rightarrow \theta^{-1}(x) = \theta(x^{\lambda}) \tag{2.12}$$

for all $x \in A$.
Using (2.12) and put $a = x^{\rho}, b = y^{\rho}$ in equation (2.11), we have

$$wk \cdot k\theta(x)((k\theta(x)\backslash(k\backslash v))k\theta(x)\theta(y)\theta(x)) = (wv)k\theta(x)\theta(y)\theta(x) \tag{2.13}$$

for all $x,\ y \in A$ and all $u,\ w,\ k \in K$ and the rest is clear. □

Lemma 2.9. *Let $Q,\ A,$ and K be loops such that $Q = A \times_\theta K$, where $\theta : A \to SymK$. Then Q is an RWPC loop if and only if*
(i) A is an RWPC loop,

(ii) $\theta(x)$ is an automorphism of K, for all $x,\ y \in A$,

(iii) $B(k,\ x,\ y) = (R(k),\ L^{-1}(k)L^{-1}(k\theta(x))R(k\theta(x)\theta(y)\theta(x))L(k\theta(x)),\ R(k\theta(x)\theta(y)\theta(x)))$ is an autotopism of K for all $x,\ y \in A$, all $k \in K$.

Proof. By Lemma2.7 and Lemma 2.8, we only need to prove that $\theta(x)$ is an automorphism of K.
Assuming that Q is an RWPC loop. Then by Lemma 2.7, equation (2.8) holds for all $a_1,\ a_2,\ a_3 \in A$, all $k_1,\ k_2,\ k_3 \in K$ in .
Setting $k_1 = e_K$ and $a_2 = e_A$ in equation (2.8), we obtain

$$(k_3\theta(a_1) \cdot \theta(a_1^2) \cdot k_2\theta(a_1) = (k_3\theta(a_1^2)k_2)\theta(a_1) \tag{2.14}$$

for all $a_1a_2 \in A$, all $k_2,\ k_3 \in K$.
Setting $k_2 = e_K$ in (2.14), we get

$$k_3(\theta(a_1)\theta(a_1^2) = k_3\theta(a_1^2)\theta(a_1), \tag{2.15}$$

substituting (2.15) in (2.14) and replacing $k_3\theta(a_1^2)$ by k, yields

$$k\theta(a_1) \cdot k_2\theta(a_1) = (kk_2)\theta(a_1).$$

□

Definition 2.10. An automorphism θ of a loop Q is right nuclear if $x^{-1} \cdot x\theta \in N_\rho(Q)$ for all $x \in Q$.

Theorem 2.11. *Let $Q,\ A,$ and K be loops such that $Q = A \times_\theta K$, where $\theta : A \to SymK$. Then Q is an RWPC loop if and only if*
(i) A and K are RWPC loops,
(ii) $\theta(x)$ is a right nuclear automorphism of K, for each $x \in A$.

Proof. (i) Suppose Q is an RWPC loop. Then by Theorem 1.2, A and K are RWPC loops since they are isomorphic to subloops of Q. So (i) holds by 2.9.

(ii) Now, from Lemma 2.8, $B(k,\ x,\ y)$ is an autotopism of K.
Applying the autotopism $B(k,\ x,\ y)$ to the product wv yields

$$wk\cdot k\theta(x)((k\theta(x)\backslash(k\backslash v))k\theta(x)\theta(y)\theta(x))=(wv)k\theta(x)\theta(y)\theta(x),$$

set $x=y=e_A$,

$$wk\cdot k(k\backslash(k\backslash v))k)=(wv)k.$$

Thus, $B(k,\ e_A,\ e_A)=(R(k),\ L^{-1}(k)L^{-1}(k)R(k)L(k),\ R(k))$ is an autotopism of K.
Similarly,

$$B(k,\ e_A,\ y)=(R(k),\ L^{-1}(k)L^{-1}(k)Rk(\theta(y)L(k),\ Rk(\theta(y))$$

is an autotopism of K.
Now

$$\begin{aligned}&B(k,\ e_A,\ e_A)^{-1}B(k,\ e_A,\ y)\\&=(R(k),\ L^{-1}(k)L^{-1}(k)R(k)L(k),\ R(k))^{-1}(R(k),\ L^{-1}(k)L^{-1}(k)Rk(\theta(y)L(k),\ Rk(\theta(y)),\\&=(I,\ L^{-1}(k)R^{-1}(k)L^{2}(k)L^{-1}(k)L^{-1}(k))Rk\theta(y)L(k),\ R^{-1}(k)Rk(\theta(y)).\quad(2.16)\end{aligned}$$

Applying the autotopism in (2.16) to the product tz and let

$$U=L^{-1}(k)R^{-1}(k)L^{2}(k)L^{-1}(k)L^{-1}(k))Rk\theta(y)L(k),$$

we have

$$t\cdot zU=(tz)R^{-1}(k)Rk(\theta(y).\tag{2.17}$$

Setting $t=e_A$, we see that $U=R^{-1}(k)Rk(\theta(y)$.
The autotopism in (2.16) now becomes

$$(I,\ R^{-1}(k)Rk(\theta(y),\ R^{-1}(k)Rk(\theta(y)).\tag{2.18}$$

Applying the autotopism in (2.18) to the product xz yields

$$x\cdot zR^{-1}(k)Rk\theta(y)=(xz)R^{-1}(k)Rk\theta(y),$$

and with $z=e_A$, we have

$$R^{-1}(k)\cdot R(k\theta(y))=R(k^{-1}\cdot k\theta(y))$$

for all $y\in A$, all $k\in K$.
So the autotopism in (2.18) becomes

$$(I,\ R(k^{-1}\cdot k\theta(y)),\ R(k^{-1}\cdot k\theta(y))).$$

Thus, $k^{-1}\cdot k\theta(y)\in N_\rho$.
Conversely, suppose (i) and (ii) hold. Then $k^{-1}\cdot k\theta(y)\in N_\rho$.

Let $k^{-1}\cdot k\theta(y) = p$, for $p \in N_\rho(K)$.
Now,

$$B(k,\ e_A,\ e_A)^{-1}B(k,\ e_A,\ y) = (I,\ R^{-1}(k)R(k\theta(y)),\ R^{-1}(k)R(k\theta(y))),$$
$$\Rightarrow t\cdot zR^{-1}(k)R(k\theta(y)) = (tz)R^{-1}(k)R(k\theta(y)),$$
$$\Rightarrow tR(k^{\lambda}\cdot k\theta(y)) = tR^{-1}(k)R(k\theta(y)),$$
$$\Rightarrow tR(k^{-1}\cdot k\theta(y)) = tR^{-1}(k)R(k\theta(y)).$$

Thus, $B(k,\ e_A,\ e_A)^{-1}B(k,\ e_A,\ y) = (I,\ R(k^{-1}\cdot k\theta(y)),\ R(k^{-1}\cdot k\theta(y))) = (I,\ R(p),\ R(p))$ is an autotopism of K and Q is an RWPC loop by Lemma 2.9. □

Corollary 2.12. *Let Q, A, and K be loops with $Q = A\times_\theta K$, where $\theta : A \to SymK$. Then Q is a WIP PACC loop if and only if*
(i) A and K are WIP PACC loops,

(ii) $\theta(x)$ is a nuclear automorphism of K, for each $x \in A$,

(iii) $\theta(xy) = \theta(x)\theta(y)$.

(iv) $A(k,\ x,\ y) = (R^{-1}(k)R^{-1}(k\theta(x))L(k\theta(y)\theta(x)\theta(x))R(k\theta(x)),\ L(k),\ L(k\theta(y)\theta(x)\theta(x)))$ *and*
$B(k,\ x,\ y) = (R(k),\ L^{-1}(k)L^{-1}(k\theta(x))R(k\theta(x)\theta(y)\theta(x))L(k\theta(x)),\ R(k\theta(x)\theta(y)\theta(x)))$
are autotopism of Q.

Corollary 2.13. *Let Q, A, and K be loops with $Q = A\times_\theta K$, where $\theta : A \to SymK$. Then Q is an extra loop if and only if*
(i) A and K are extra loops,

(ii) $\theta(x)$ is a nuclear automorphism of K, for each $x \in A$.

Example 2.14. Here is a WIP PACC loop of order 16 with $N(Q) = \{1, 2, 3, 4\}$.

TABLE 3. AWIP PACC loop of order 16

$\star$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	8	7	6	5	12	11	10	9	15	16	13	14
4	4	3	2	1	7	8	5	6	11	12	9	10	16	15	14	13
5	5	6	7	8	1	2	3	4	15	16	13	14	12	11	10	9
6	6	5	8	7	2	1	4	3	16	15	14	13	11	12	9	10
7	7	8	5	6	4	3	2	1	14	13	16	15	10	9	12	11
8	8	7	6	5	3	4	1	2	13	14	15	16	9	10	11	12
9	9	10	11	12	13	14	15	16	1	2	3	4	6	5	8	7
10	10	9	12	11	14	13	16	15	2	1	4	3	5	6	7	8
11	11	12	9	10	16	15	14	13	4	3	2	1	8	7	6	5
12	12	11	10	9	15	16	13	14	3	4	1	2	7	8	5	6
13	13	14	15	16	10	9	12	11	7	8	5	6	4	3	2	1
14	14	13	16	15	9	10	11	12	8	7	6	5	3	4	1	2
15	15	16	13	14	11	12	9	10	6	5	8	7	2	1	4	3
16	16	15	14	13	12	11	10	9	5	6	7	8	1	2	3	4

REFERENCES

1. R. H. Bruck, *A survey of Binary Systems*, Springer-Verlag, Berlin-Gottingen-Heidelberg, 1958.
2. A. Drapal, and P. Jedlicka, *On loop identities that can be obtained by a nuclear identification,* European Journal of Combinatorics, **31** (2010), 1907-1923.
3. O. O. George, J. O. Olaleru, J. O. Adéníran, T. G. Jaiyéọlá, *On a class of power associative LCC-loops*, (2022) Extracta Mathematicae. Retrieved from https://publicaciones.unex.es/index.php/EM/article/view/1201
4. O. O. George and T. G. Jaiyéọlá, *Nuclear Identificatoion of Some New Loop Identities of Lenght Five*, Pre-Print.
5. E. G. Goodaire and D. A. Robinson, *Semi-Direct Products and Bol Loop,* Demonstratio Mathematica. **27** (1994), 662-672.
6. M.K. Kinyon, K. Kunen, and J. D.Phillips, *Diassociativity in conjugacy closed loops,* comm. Algebra, **32** (2004), 767-786.
7. Nagy G. P. and Vojtěchovský, P., *The LOOPS Package, Computing with quasigroups and loops in GAP 3.4.1*, http://www.math.du.edu/loops.
8. The GAP Group, *GAPS - Groups, Algorithms, Programming, Version 4.11.0*, http://www.gap-system.org

9. H. O. Pflugfelder, *Quasigroups and loops: Introduction,* Sigma Series in Pure Math. **7,** Heldermann Verlag, Berlin, 1990.
10. J. D. Phillips, *A short basis for the variety of WIP PACC- loops,* Quasigroups and Related Systems, **14** (2006), 259-271.

[1] Department of Mathematics, University of Lagos, Akoka, Nigeria..
Email address: oogeorge@unilag.edu.ng