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CONCERNING MULTIPLICATIVE (GENERALIZED) (α, β)-DERIVATIONS IN PRIME RINGS

ABU ZAID ANSARI*

ABSTRACT. The main objective of this scientific research is to study some algebraic identities with multiplicative (generalized) (α, β)-derivations on the left ideal of a prime ring R.

1. INTRODUCTION AND PRELIMINARIES

Throughout R will represent an associative ring. A ring R is n-torsion free,if $nx = 0$ implies $x = 0$ for all $x \in R$, where $n > 1$ is an integer. A ring R is called prime ring if $aRb = (0)$ implies either $a = 0$ or $b = 0$. An additive mapping $\delta : R \to R$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all pairs $x, y \in R$ and is said to be a Jordan derivation if $\delta(x^2) = \delta(x)x + x\delta(x)$ is fulfilled for all $x \in R$. Following [1], an additive mapping $F : R \to R$ is said to be a generalized derivation if there exists an associated derivation $\delta : R \to R$ such that $F(xy) = F(x)y + x\delta(y)$ holds for all pairs $x, y \in R$. Of course a generalized derivation is a generalization of derivation. Inspired by Martindale [8], in 1991, Daif [2] has given a generalization of derivation as multiplicative derivation which is defined as: a mapping $d : R \to R$ (not necessarily additive) is said to be multiplicative derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Similar type of notation is defined in [7]. Later, in [3] Daif and Tammam have extended this notation to multiplicative generalized derivation as follows: a mapping $G : R \to R$ is a multiplicative generalized derivation if there exists a derivation g such that $G(xy) = G(x)y + xg(y)$ for all $x, y \in R$. If we consider g is any map neither necessarily derivative nor additive, then G is called multiplicative (generalized)-derivation which was introduced by Dhara and Ali in [4]. The concept of multiplicative (generalized)-derivation covers the concept of multiplicative generalized derivation and multiplicative centralizer (if $g = 0$). A mapping f from R to R is centralizing if $[f(x), x] \in Z(R)$ and particularly commuting if $[f(x), x] = 0$ for all $x \in R$. Posner [9] has initiated to investigate the commutativity of rings with derivations. In details, he proved that a prime ring R is commutative if there is a non zero derivation δ which is centralizing on R. Several authors extended this

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* Corresponding author.
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result in different directions (see [5, 6, 11]). Further, Dhara and Ali considered the following identities with multiplicative (generalized)-derivation in semiprime ring: (i) $F(xy) \pm xy = 0$, (ii) $F(xy) \pm yx = 0$, (iii) $F(x)F(y) \pm xy \in Z(R)$, (iv) $F(xy) \pm yx \in Z(R)$ for all x and y in suitable subset of R. Then, this type of work is generalized by Rehman et al. [10] by introducing a new concept of multiplicative (generalized) (α,β)-derivations in prime rings. Motivated by the work in [10], we extend some results in the present paper.

2. Main results

Let us begin with the following theorem:

Theorem 2.1. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $G, D : R \to R$ are multiplicative generalized (α,β)-derivations associated with the maps g, d respectively. If G and D satisfy $G(xy) = D(xy)$ for all $x, y \in I$, then G and D are identical on I, where α and β are automorphisms on R.*

Proof. We have

$$G(xy) = D(xy) \text{ for all } x, y \in I \tag{2.1}$$

Replacing y by yz, we get $\beta(xy)[g(z) - d(z)] = 0$ for all $x, y, z \in I$. This implies that $\beta(x)\beta(y)[g(z) - d(z)] = 0$ for all $x, y, z \in I$. Replace y by ry to get $\beta(x)\beta(ry)[g(z)-d(z)] = 0$ for all $x, y, z \in I$ and $r \in R$. i.e., $\beta(x)\beta(r)\beta(y)[g(z)-d(z)] = 0$ for all $x, y, z \in I$ and $r \in R$. Replacing r by $\beta^{-1}(r)$, we obtain $\beta(x)r\beta(y)[g(z) - d(z)] = 0$ for all $x, y, z \in I$ and $r \in R$. Primeness of R implies that either $\beta(x) = 0$ or $\beta(y)[g(z) - d(z)] = 0$ for all $x, y, z \in I$. If $\beta(x) = 0$ for all $x \in I$, then $\beta(I) = (0)$. Since I is nonzero left ideal of R, then we get a contradiction. So, $\beta(y)[g(z) - d(z)] = 0$ for all $y, z \in I$, then

$$\beta(y)g(z) = \beta(y)d(z) \text{ for all } y, z \in I \tag{2.2}$$

Again, from (2.1)

$$G(x)\alpha(y) + \beta(x)g(y) \;=\; D(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in I \tag{2.3}$$

Using (2.2), we get $G(x)\alpha(y) = D(x)\alpha(y)$ for all $x, y \in I$. Replacing y by ry in the last expression, we obtain $[G(x) - D(x)]\alpha(ry)$ for all $x, y \in I$ and $r \in R$. This implies that $[G(x) - D(x)]\alpha(r)\alpha(y)$ for all $x, y \in I$ and $r \in R$. Replace r by $\alpha^{-1}(r)$ to obtain $[G(x) - D(x)]r\alpha(y)$ for all $x, y \in I$ and $r \in R$. Using primeness of R, we get either $\alpha(y) = 0$ or $G(x) = D(x)$ for all $x, y \in I$. Again, $\alpha(y) = 0$ gives a contradiction, hence $G = D$ on I, which is the required result.

Theorem 2.2. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $G, D : R \to R$ are multiplicative generalized (α,β)-derivations associated with the maps g, d respectively. If G and D satisfy $G(x)D(y) = \alpha(xy)$ for all $x, y \in I$, then $\beta(I)d(I) = (0)$, where α and β are automorphisms on R.*

Proof. Given that

$$G(x)D(y) = \alpha(xy) \text{ for all } x, y, z \in I. \tag{2.4}$$

Replacing y by yz, we obtain

$$G(x)\beta(y)d(z) = 0 \text{ for all } x, y, z \in I. \tag{2.5}$$

Replace y by ry to get

$$G(x)\beta(r)\beta(y)d(z) = 0 \text{ for all } x, y, z \in I \text{ and } r \in R. \tag{2.6}$$

Now, replacing r by $\beta^{-1}(r)$, we find that

$$G(x)r\beta(y)d(z) = 0 \text{ for all } x, y, z \in I \text{ and } r \in R. \tag{2.7}$$

Since R is prime ring, hence either $G(x) = 0$ or $\beta(y)d(z) = 0$ for all $x, y, z \in I$. If $G(x) = 0$ for all $x \in I$, then $\alpha(xy) = 0$ for all $x, y \in I$. Replacing y by ry, we get $\alpha(x)\alpha(r)\alpha(y) = 0$ for all $x, y \in I$ and $r \in R$. Replace r by $\alpha^{-1}(r)$ to find $\alpha(x)r\alpha(y) = 0$ for all $x, y \in I$ and $r \in R$. Using primness $\alpha(I) = (0)$, a contradiction, hence $\beta(y)d(z) = 0$ for all $y, z \in I$. In other words $\beta(I)d(I) = 0$.

Theorem 2.3. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $G, D : R \to R$ are multiplicative generalized (α, β)-derivations associated with the map g, d respectively. If G and D satisfy $G(xy) = G(x)D(y)$ for all $x, y \in I$, then either $G(xy) = G(x)\alpha(y)$ or $D(xy) = \beta(x)D(y)$ for all $x, y \in I$, where α and β are automorphisms on R.*

Proof. Given that

$$G(xy) = G(x)D(y) \text{ for all } x, y \in I. \tag{2.8}$$

Replacing y by yz, we find that

$$\beta(xy)g(z) = G(x)\beta(y)d(z) \text{ for all } x, y, z \in I. \tag{2.9}$$

Again, replace x by xw to get

$$\beta(xwy)g(z) = G(x)\alpha(w)\beta(y)d(z) + \beta(x)g(w)\beta(y)d(z) \text{ for all } x, y, z \in I. \tag{2.10}$$

Replacing y by wy in (2.9), we get

$$\beta(xwy)g(z) = G(x)\beta(w)\beta(y)d(z) \text{ for all } x, y, z, w \in I. \tag{2.11}$$

Now, equating (2.10) and (2.11), we obtain

$$[G(x)\alpha(w) + \beta(x)g(w) - G(x)\beta(w)]\beta(y)d(z) = 0 \text{ for all } x, y, z, w \in I.$$

This implies that $[G(xw) - G(x)\beta(w)]\beta(y)d(z) = 0$ for all $x, w \in I$. Replacing r by ry, we get $[G(xw) - G(x)\beta(w)]\beta(r)\beta(y)d(z) = 0$ for all $x, y, z, w \in I$ and $r \in R$. Replace r by $\beta^{-1}(r)$ to obtain $[G(xw) - G(x)\beta(w)]r\beta(y)d(z) = 0$ for all $x, w \in I$. Primeness of R forces that either $G(xw) = G(x)\beta(w)$ or $\beta(y)d(z) = 0$ for all $x, y, z, w \in I$. If $\beta(y)d(z) = 0$, then $D(xy) = D(x)\alpha(y)$ for all $x, y \in I$.

Theorem 2.4. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $G, D : R \to R$ are multiplicative generalized (α, β)-derivations associated with the maps g, d respectively. If G and D satisfy $G(xy) = D(x)G(y)$ for all $x, y \in I$, then either $G(xy) = G(x)\alpha(y)$ or $D(xy) = \beta(x)D(y)$ for all $x, y \in I$, where α and β are automorphism on R.*

Proof. Given that

$$G(xy) = D(x)G(y) \text{ for all } x, y \in I. \tag{2.12}$$

Replacing y by yz, we find that

$$\beta(xy)g(z) = D(x)\beta(y)g(z) \text{ for all } x, y, z \in I. \tag{2.13}$$

Again, replace x by xw to get

$$\beta(xwy)g(z) = D(x)\alpha(w)\beta(y)g(z) + \beta(x)d(w)\beta(y)g(z) \text{ for all } x, y, z \in I. \tag{2.14}$$

Replacing y by wy in (2.13), we get

$$\beta(xwy)g(z) = D(x)\beta(w)\beta(y)g(z) \text{ for all } x, y, z \in I. \tag{2.15}$$

Now, equating (2.14) and (2.15), we obtain

$$[D(x)\alpha(w) + \beta(x)d(w) - D(x)\beta(w)]\beta(y)g(z) = 0$$

for all $x, y, z, w \in I$. This implies that $[D(xw) - D(x)\beta(w)]\beta(y)g(z) = 0$ for all $x, y, z, w \in I$. Replacing y by ry, we get $[D(xw) - D(x)\beta(w)]\beta(r)\beta(y)g(z) = 0$ for all $x, y, , z, w \in I$ and $r \in R$. Replace r by $\beta^{-1}(r)$ to obtain $[D(xw) - D(x)\beta(w)]r\beta(y)g(z) = 0$ for all $x, y, z, w \in I$ and $r \in R$. Primeness of R forces that either $D(xw) = D(x)\beta(w)$ or $\beta(y)g(z) = 0$ for all $x, y, z, w \in I$. If $\beta(y)g(z) = 0$, then $G(xy) = G(x)\alpha(y)$ for all $x, y \in I$, which is the required result.

The following example is in favour of the theorem:

Example 2.5. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}$. Define $G, D, g, d, \alpha, \beta : R \to R$ by $G \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & -c \end{pmatrix}$, $D \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $g \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$ and $\beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. It is easy to check that I is a left ideal of R and $G,\ D$ are multiplicative (generalized) (α, β)-derivations and $G,\ D$ satisfy the condition $G(xy) = D(xy)$ for all $x, y \in I$ but $G(x) \neq D(x)$ for all $x \in I$. Hence, the primeness hypothesis in Theorem 2.1 is crucial.

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Department of Mathematics, Faculty of Science, Islamic University of Madinah, KSA
Email address: ansari.abuzaid@gmail.com, ansari.abuzaid@iu.edu.sa