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GROWTH AND OSCILLATION OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS OF [P,Q]-ORDER IN THE UNIT DISC

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Abstract. This paper is devoted to study the properties of solutions of linear differential equations when the coefficients are meromorphic functions in the unit disc. By comparing the growth conditions on the coefficients in the term of $[p,q]$ −order near a point on the boundary of the unit disc, we obtain general results about the growth and the oscillation of the solutions of such equations. We improve some recent results of Hamouda [8], Chen-Deng [6], Qin, Long and Li [15].

1. Introduction and main results

In this paper, we are particularly interested in the study of the growth and oscillation properties of solutions of the linear differential equation of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_i(z)$ $(i = 0, 1, \ldots, k-1)$, $k \geq 2$ are analytic or meromorphic in the unit disc $\Delta = \{z : |z| < 1\}$. The growth and oscillation of solutions of linear differential equation (1.1) has been studied by different researchers in the unit disc by making use of Nevanlinna's theory [1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15, 17, 18]. Note that in [9, 10, 13, 16] there are some fundamental concepts of the Nevanlinna's theory in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. This paper is mainly concerned with the $[p,q]$-order of solutions of (1.1) which is the generalization of the paper of Hamouda [8] who characterized the iterated n-order of solutions of (1.1) and the improvement of the results of Chen and Deng [6], Qin *et al.* [15]. In fact, the growth conditions on the coefficients will be expressed by using the $[p,q]$-order instead of the iterated n-order as defined in the paper of Hamouda [8] and the papers of Chen and Deng [6], Qin *et al.* [15].

To generalize some results on the properties of solutions of some differential equations, we need to define the $[p,q]$-order of a meromorphic function, but first

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we need to define the following expressions on the exponential and its reciprocal function : for all $r \in \mathbb{R}$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, +\infty)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1. [4] Let f be a meromorphic function in Δ. Then the iterated $p-$order of f is defined by

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,\ f)}{\log \frac{1}{1-r}} \quad (p \geq 1 \text{ is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. If f is analytic in Δ, then the iterated $p-$order of f is defined by

$$\rho_{M,p}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,\ f)}{\log \frac{1}{1-r}} \quad (p \geq 1 \text{ is an integer}).$$

Remark 1.2. It follows by M. Tsuji ([16], p. 205) that if f is an analytic function in Δ, then we have the inequalities

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1$$

which are the best possible in the sense that there are analytic functions g and h such that $\rho_{M,1}(g) = \rho_1(g)$ and $\rho_{M,1}(h) = \rho_1(h) + 1$, see [7]. However, it follows by Theorem 1.6 in ([9], p. 18) that $\rho_{M,p}(f) = \rho_p(f)$ for $p \geq 2$.

Now, we introduce the concept of $[p, q]$-order of meromorphic and analytic functions in the unit disc. In the whole suite, we suppose p and q are integers satisfying $p \geq q \geq 1$.

Definition 1.3. [1] Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in Δ, the $[p, q]$-order of f is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r, f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log_q \frac{1}{1-r}}.$$

Remark 1.4. It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. If f is non-admissible, i.e., $T(r, f) = O\left(\log \frac{1}{1-r}\right)$, then $\rho_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.3, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$, $\rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p+1,1]}(f) = \rho_{p+1}(f)$.

Proposition 1.5. [1] *Let $p \geq q \geq 1$ be integers, and let f be an analytic function in Δ of $[p, q]$-order. The following two statements hold:*
(i) If $p = q$, then

$$\rho_{[p,q]}(f) \leq \rho_{M,[p,q]}(f) \leq \rho_{[p,q]}(f) + 1.$$

(ii) If $p > q$, then

$$\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f).$$

Definition 1.6. [14] Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in Δ. The $[p,q]$-exponent of convergence of the zero sequence of f is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log_q \frac{1}{1-r}},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of f in $\{z : |z| \leq r\}$. Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log_q \frac{1}{1-r}},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z : |z| \leq r\}$.

Definition 1.7. Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in Δ. The $[p,q]$-exponent of convergence of the sequence of poles of f is defined by

$$\lambda_{[p,q]}\left(\frac{1}{f}\right) = \limsup_{r \to 1^-} \frac{\log_p N(r,f)}{\log_q \frac{1}{1-r}},$$

where $N(r,f)$ is the integrated counting function of poles of f in $\{z : |z| \leq r\}$.

In [8], Hamouda studied the growth of solutions of equation (1.1) when A_0 dominates the other coefficients near the unit disc's boundary to find the properties of fast growing solutions of (1.1) and he obtained the following two results that improve and generalize the results of Heittokangas *et al.* [11].

Theorem 1.8. [8] *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be meromorphic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 such that for any constant $\mu > 0$, we have*

$$\lim_{\substack{|z| \to 1^- \\ z \in \gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \left(\frac{1}{(1-|z|)^\mu}\right) = 0,$$

then every nontrivial meromorphic solution f of equation (1.1) is of infinite order.

Theorem 1.9. [8] *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be meromorphic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 such that for any constant $\mu > 0$, we have*

$$\lim_{\substack{|z| \to 1^- \\ z \in \gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_n\left\{\frac{\lambda}{(1-|z|)^\mu}\right\} = 0,$$

where $n \geq 1$ is an integer, $\lambda > 0$ is a real constant, then every nontrivial meromorphic solution f of equation (1.1) satisfies $\rho_n(f) = \infty$ and $\rho_{n+1}(f) \geq \mu$.

Recently, in the study of the growth of solutions of equation (1.1), Qin *et al.* [15] improved the result of Hamouda [8] in the case when the coefficients of equation (1.1) are analytic functions and obtained the following result.

Theorem 1.10. [15] *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 satisfying*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\frac{1}{1-|z|} \right)^{\mu} \right\} < 1,$$

where $\mu > 0$ and $\lambda > 0$ are two real constants, then every nontrivial solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Also, Qin *et al.* [15] treated the case when A_s, $s = 1, ..., k-1$, $k \geq 2$ dominates the other coefficients near the unit disc's boundary and got this result.

Theorem 1.11. [15] *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ and there exists $\mu \in (0, +\infty)$ satisfying*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., s-1, s+1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_s) = \mu.$$

If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and a real constant $\lambda > 0$, such that

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p \left\{ \frac{\lambda}{(1-|z|)^{\mu}} \right\} < 1.$$

Then every nontrivial solution f of equation (1.1) in which $f^{(n)}(z)$ just has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p,q]}(A_s)$.

In this paper, we will express the growth conditions on the coefficients which are meromorphic functions of $[p,q]$ $-$order instead of analytic functions of iterated n-order as defined in the paper of Qin *et al.* [15].

Theorem 1.12. *Let $p \geq q \geq 1$ be integers. Suppose that $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ are meromorphic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and two real constants $\mu > 0$ and $\lambda > 0$ such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\} < 1, \tag{1.2}$$

then every nontrivial meromorphic solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Corollary 1.13. *Let $p \geq q \geq 1$ be integers. Suppose that $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ are meromorphic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and three real constants $\mu > 0$, $0 \leq \beta < \alpha$ such that for $z \in \gamma$ and $|z| \to 1^-$, we have*

$$|A_0(z)| \geq \exp_p \left\{ \alpha \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\}$$

and

$$|A_i(z)| \leq \exp_p \left\{ \beta \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\}, \quad i = 1, ..., k-1,$$

then every nontrivial meromorphic solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Proof. In fact, by the assumptions, we have

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\}$$
$$\leq \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{(k-1)\exp_p \left\{ \beta \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\} + 1}{\exp_p \left\{ \alpha \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\}}$$
$$\times \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\}.$$

We pose $X = \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \to +\infty$, $|z| \to 1^-$, so by taking $0 < \lambda < \alpha - \beta$, we obtain

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\}$$
$$\leq \lim_{X\to+\infty} \frac{(k-1)\exp_p\{\beta X\} + 1}{\exp_p\{\alpha X\}} \exp_p\{\lambda X\}$$
$$= \lim_{X\to+\infty} (k-1)\exp\left\{\exp_{p-1}\{\beta X\} - \exp_{p-1}\{\alpha X\} + \exp_{p-1}\{\lambda X\}\right\}$$
$$+ \exp\left\{\exp_{p-1}\{\lambda X\} - \exp_{p-1}\{\alpha X\}\right\} = 0,$$

then we get

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-|z|} \right) \right)^{\mu} \right\} = 0 < 1.$$

Hence, by Theorem 1.12, we obtain $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$. $\square$

By using Corollary 1.13 and Lemma 2.7, we easily obtain the following corollary.

Corollary 1.14. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ and for $\mu \in (0, +\infty)$ we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_0) = \mu.$$

If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and some real constants $\mu > 0$, $0 \leq \beta < \alpha$ such that for $z \in \gamma$ and $|z| \to 1^-$, we have

$$|A_0(z)| \geq \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$|A_i(z)| \leq \exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ i = 1, ..., k-1,$$

then every nontrivial solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_0) = \mu$.

Theorem 1.15. *Let $p \geq q \geq 1$ be integers. Suppose that $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ are meromorphic functions in Δ and there exist $w_0 \in \partial\Delta$, a curve $\gamma \subset \Delta$ tending to w_0 and $\mu > 0$ such that $\lambda_{[p,q]}\left(\frac{1}{A_0}\right) < \mu$ and*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1. \qquad (1.3)$$

Then every nontrivial meromorphic solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Corollary 1.16. *Let $p \geq q \geq 1$ be integers. Suppose that $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ are meromorphic functions in Δ and there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and some real constants $\mu > 0$ and $0 \leq \beta < \alpha$ such that $\lambda_{[p,q]}\left(\frac{1}{A_0}\right) < \mu$ and*

$$T(r, A_0) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},$$

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\},\ i = 1, ..., k-1,$$

where $z \in \gamma$ and $|z| \to 1^-$. Then the following statements hold:
(i) If $p = q = 1$ and $0 \leq (k-1)\beta < \alpha$, then every nontrivial meromorphic solution f of equation (1.1) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \mu$.
(ii) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every nontrivial meromorphic solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

Proof. In fact, by the assumptions, we have

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$\leq \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} \exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}}{\exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$= \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\left(\exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}\right)^{k-1}}{\exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}.$$

If $p = q = 1$, then for $0 < \lambda < \alpha - (k-1)\beta$, we have

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{m(r,A_0)}} \exp\left\{\lambda\left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

$$\leq \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\left(\exp\left\{\beta\left(\frac{1}{1-|z|}\right)^{\mu}\right\}\right)^{k-1}}{\exp\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\mu}\right\}} \exp\left\{\lambda\left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

$$= \lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \exp\left\{\left((k-1)\beta - \alpha + \lambda\right)\left(\frac{1}{1-|z|}\right)^{\mu}\right\}$$

so in this case we get

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp\left\{\lambda\left(\frac{1}{1-|z|}\right)^{\mu}\right\} = 0 < 1.$$

If $p \geq q \geq 2$, then for $0 < \lambda < \alpha - \beta$, by setting $X = \left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu} \to +\infty$, $|z| \to 1^-$, we obtain

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$\leq \lim_{X\to+\infty} \frac{\left(\exp_p\{\beta X\}\right)^{k-1}}{\exp_p\{\alpha X\}} \exp_p\{\lambda X\}$$

$$= \lim_{X\to+\infty} \frac{\exp\left\{(k-1)\exp_{p-1}\{\beta X\}\right\}}{\exp\left\{\exp_{p-1}\{\alpha X\}\right\}} \exp\left\{\exp_{p-1}\{\lambda X\}\right\}$$

$$= \lim_{X\to+\infty} \exp\left\{(k-1)\exp_{p-1}\{\beta X\} + \exp_{p-1}\{\lambda X\} - \exp_{p-1}\{\alpha X\}\right\}$$

$$= \lim_{X\to+\infty} \exp\left\{\exp_{p-1}\{\alpha X\}\left((k-1)\frac{\exp_{p-1}\{\beta X\}}{\exp_{p-1}\{\alpha X\}} + \frac{\exp_{p-1}\{\lambda X\}}{\exp_{p-1}\{\alpha X\}} - 1\right)\right\} = 0,$$

so, we obtain

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} = 0 < 1.$$

Hence, by applying Theorem 1.15 we get Corollary 1.16. □

By using Theorem 1.15 and Lemma 2.7, we easily obtain the following corollary.

Corollary 1.17. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ and for $\mu \in (0, +\infty)$ we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_0) = \mu.$$

Suppose there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and a real constant $\lambda > 0$ such that

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1.$$

Then every nontrivial solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_0) = \mu$.

By using Corollary 1.16 and Lemma 2.7, we easily obtain the following corollary.

Corollary 1.18. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ and for $\mu \in (0, +\infty)$ we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_0) = \mu.$$

Suppose there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and some constants $0 \leq \beta < \alpha$, such that

$$T(r, A_0) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, \ i = 1, ..., k-1,$$

where $z \in \gamma$ and $|z| \to 1^-$. Then the following statements hold:
(i) If $p = q = 1$ and $0 \leq (k-1)\beta < \alpha$, then every nontrivial solution f of equation (1.1) satisfies $\rho_1(f) = \rho_{M,1}(f) = \infty$ and $\rho_2(f) = \rho_{M,2}(f) = \rho_{M,2}(A_0)$.
(ii) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every nontrivial solution f of equation (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_0) = \mu$.

Theorem 1.19. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be meromorphic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and real constants $\lambda > 0$ and $\mu > 0$ such that $\lambda_{[p,q]}\left(\frac{1}{A_s}\right) < \mu$ and*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1. \tag{1.4}$$

Then every nontrivial meromorphic solution f of equation (1.1), such that $f^{(n)}(z)$ has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

By applying Theorem 1.19, we get the following corollary.

Corollary 1.20. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be meromorphic functions in Δ. If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and real constants $\mu > 0$ and $0 \leq \beta < \alpha$ such that $\lambda_{[p,q]}\left(\frac{1}{A_s}\right) < \mu$ and*

$$T(r, A_s) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, i = 1, ..., s-1, s+1, ..., k-1,$$

where $z \in \gamma$ and $|z| \to 1^-$. Then the following statements hold:
(i) If $p = q = 1$ and $0 \leq (k-1)\beta < \alpha$, then every nontrivial meromorphic solution f of equation (1.1), such that $f^{(n)}(z)$ has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \mu$.
(ii) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every nontrivial meromorphic solution f of equation (1.1), such that $f^{(n)}(z)$ has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho_{[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) \geq \mu$.

By using Theorem 1.19 and Lemma 2.7, we easily obtain the following corollary.

Corollary 1.21. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ and for any constant $\mu \in (0, +\infty)$, we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., s-1, s+1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_s) = \mu.$$

If there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and a real constant $\lambda > 0$, such that

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} < 1.$$

Then every nontrivial solution f of equation (1.1), such that $f^{(n)}(z)$ has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_s) = \mu$.

By using Corollary 1.20 and Lemma 2.7, we easily obtain the following corollary.

Corollary 1.22. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be analytic functions in Δ and for any constant $\mu \in (0, +\infty)$, we have*

$$\max\left\{\rho_{M,[p,q]}(A_i) : i = 1, ..., s-1, s+1, ..., k-1\right\} \leq \rho_{M,[p,q]}(A_s) = \mu.$$

Suppose there exist $w_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to w_0 and real constants $0 \leq \beta < \alpha$, such that

$$T(r, A_s) \geq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, i = 1, ..., s-1, s+1, ..., k-1,$$

where $z \in \gamma$ and $|z| \to 1^-$. Then the following statements hold:
(i) If $p = q = 1$ and $0 \leq (k-1)\beta < \alpha$, then every nontrivial solution f of equation (1.1), such that $f^{(n)}(z)$ has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho_1(f) = \rho_{M,1}(f) = \infty$ and $\rho_2(f) = \rho_{M,2}(f) = \rho_M(A_s)$.
(ii) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every nontrivial solution f of equation (1.1), such that $f^{(n)}(z)$ has finite many zeros for all $n < s$, $(n = 0, ..., s-1)$ satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_s) = \mu$.

The second aim of this paper is to investigate the fixed points of solutions of equation (1.1). We obtain the following result.

Theorem 1.23. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ $(k \geq 2)$ be meromorphic functions in Δ. Suppose that all solutions of the equation (1.1) are of infinite $[p,q]$-order and $\rho_{[p+1,q]}(f) = \rho < +\infty$. If $\varphi(z) \not\equiv 0$ is a meromorphic function with $\rho_{[p,q]}(\varphi) < \infty$, then every meromorphic solution $f(z) \not\equiv 0$ of equation (1.1) satisfies*

$$\overline{\lambda}_{[p,q]}(f - \varphi) = \lambda_{[p,q]}(f - \varphi) = \rho_{[p,q]}(f) = +\infty,$$
$$\overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi) = \rho_{[p+1,q]}(f) = \rho.$$

By setting $\varphi(z) = z$, and using Theorem 1.23, we easily obtain the following corollary.

Corollary 1.24. *Assume that the assumptions of one of Corollaries 1.14, 1.17 and 1.21 holds. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies*

$$\overline{\lambda}_{[p,q]}(f - z) = \lambda_{[p,q]}(f - z) = \rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$$

and

$$\overline{\lambda}_{[p+1,q]}(f-z)=\lambda_{[p+1,q]}(f-z)=\rho_{[p+1,q]}(f)=\rho_{M,[p+1,q]}(f)=\mu.$$

2. Some Preliminary lemmas

In this section we give some lemmas which are used in the proofs of our theorems.

Lemma 2.1. *Let k and j be integers satisfying $k>j\geq 0$, and let $\varepsilon>0$ and $d\in(0,1)$. If f is a meromorphic function in Δ such that $f^{(j)}(z)$ does not vanish identically, then* [7]

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq\left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|},T(s(|z|),f)\right\}\right]^{k-j},\quad |z|\notin E,\tag{2.1}$$

where $E\subset[0,1)$ is a set with finite logarithmic measure $\int\limits_E\frac{dr}{1-r}<\infty$ and $s(|z|)=1-d(1-|z|)$. If $\rho_{[1,1]}(f)=\rho_1(f)<\infty$, then [7]

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq\left(\frac{1}{1-|z|}\right)^{(k-j)(\rho_1(f)+2+\varepsilon)},\quad |z|\notin E\tag{2.2}$$

and if $\rho_p(f)<\infty$, for $p\geq 2$, then [7]

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq\exp_{p-1}\left\{\left(\frac{1}{1-|z|}\right)^{\rho_p(f)+\varepsilon}\right\},\quad |z|\notin E\tag{2.3}$$

Moreover, if $\rho_{[p,q]}(f)<\infty$, then for $p\geq q\geq 2$

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\rho_{[p,q]}(f)+\varepsilon}\right\},\quad |z|\notin E.\tag{2.4}$$

Proof. The estimates (2.1)-(2.3) are from the paper [7]. We prove only the estimate (2.4). We know that there exists a set $E\subset[0,1)$ with finite logarithmic measure $\int\limits_E\frac{dr}{1-r}<\infty$ such that holds the estimate

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq\left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|};T(s(|z|),f)\right\}\right]^{k-j}.$$

By using the definition of the $[p,q]$-order, for any given $\varepsilon>0$ and sufficiently large r, we have

$$T(s(|z|),f)\leq\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-s(|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{4}}\right\}$$

which implies

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leq\left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-|z|};\right.\right.$$

$$\left.\left.\left.\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-s(|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{4}}\right\}\right\}\right]^{k-j}$$

$$\leq\left[\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-s(|z|)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{2}}\right\}\right]^{k-j}$$

$$=\exp\left\{(k-j)\exp_{p-2}\left\{\left(\log_{q-1}\left(\frac{1}{d\left(1-|z|\right)}\right)\right)^{\rho_{[p,q]}(f)+\frac{\varepsilon}{2}}\right\}\right\}$$

$$\leq\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\rho_{[p,q]}(f)+\varepsilon}\right\}.$$

□

Lemma 2.2. [15] *Let $f:\Delta\to\mathbb{R}$ be a meromorphic function in Δ. If there exist a point w_0 on the boundary $\partial\Delta=\{z:|z|=1\}$ and a curve $\gamma\subset\Delta$ tending to w_0 such that*

$$\lim_{\substack{z\to w_0\\ z\in\gamma}} f(z)<1,$$

then there exists a set $E_1\subset[0,1)$ with infinite logarithmic measure $\int\limits_{E_1}\frac{dr}{1-r}=\infty$ such that we have $f(z)<1$ for all $|z|\in E_1$.

Lemma 2.3. [10] *Let f be a meromorphic function in Δ, and let $k\geq 1$ be an integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f),$$

where $S(r,f)=O\left(\log^{+}T(r,f)+\log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E_2\subset[0,1)$ with $\int\limits_{E_2}\frac{dr}{1-r}<\infty$. If f is of finite order, then

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 2.4. [1] *Let $p\geq q\geq 1$ be integers. Let f be a meromorphic function in the unit disc Δ such that $\rho_{[p,q]}(f)=\rho<\infty$, and let $k\geq 1$ be an integer. Then for any $\varepsilon>0$,*

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r outside a set $E_3\subset[0,1)$ with $\int\limits_{E_3}\frac{dr}{1-r}<\infty$.

Lemma 2.5. [14] *Let $p\geq q\geq 1$ be integers, and let f and g be nonconstant meromorphic functions of $[p,q]$-order in Δ. Then we have*

$$\rho_{[p,q]}\left(f+g\right)\leq\max\left\{\rho_{[p,q]}\left(f\right),\rho_{[p,q]}\left(g\right)\right\}$$

and

$$\rho_{[p,q]}\left(fg\right)\leq\max\left\{\rho_{[p,q]}\left(f\right),\rho_{[p,q]}\left(g\right)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, *then we obtain*

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f).$$

By using the same arguments of Lemma 3.5 in the paper ([12], p. 4), we obtain the following lemma in the case when $\rho_{[p,q]}(f) = \rho = \infty$.

Lemma 2.6. *Let* $p \geq q \geq 1$ *be integers. Let* A_j $(j = 0, ..., k-1)$, $F \not\equiv 0$ *be meromorphic functions in* Δ, *and let* f *be a solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F$$

satisfying $\max\left\{\rho_{[p,q]}(A_j) \quad (j = 0, ..., k-1), \rho_{[p,q]}(F)\right\} < \rho_{[p,q]}(f) = \rho \leq \infty$. *Then we have*

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$$

and

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f).$$

Lemma 2.7. [1] *Let* $p \geq q \geq 1$ *be integers. If* $A_0(z), ..., A_{k-1}(z)$ *are analytic functions of* $[p,q]$ $-$*order in the unit disc* D, *then every solution* $f \not\equiv 0$ *of equation* (1.1)*satisfies*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max\left\{\rho_{M,[p,q]}(A_j) : j = 0, 1, ..., k-1\right\}.$$

3. Proof of Theorems

Proof of Theorem 1.12

Proof. Assume that f is a nontrivial meromorphic solution of (1.1) with finite $[p,q]$ $-$order $\rho_{[p,q]}(f) = \rho < \infty$. According to Lemma 2.1, if $p = q = 1$, then for any given $\varepsilon > 0$, there exists a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < \infty$ such that for all z satisfying $|z| = r \notin E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\frac{1}{1-r}\right)^{j(\rho+2+\varepsilon)}, \ j = 1, ..., k \tag{3.1}$$

and if $p \geq q \geq 2$, so

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho+\varepsilon}\right\}, \ j = 1, ..., k. \tag{3.2}$$

By (1.1), we can write

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)} \tag{3.3}$$

so

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.4}$$

Combining (3.1) and (3.4), we obtain for $p=q=1$

$$|A_0(z)| \leq \left(1+\sum_{i=1}^{k-1}|A_i(z)|\right)\left(\frac{1}{1-r}\right)^{k(\rho+2+\varepsilon)} \tag{3.5}$$

and combining (3.2) and (3.4), we get for $p \geq q \geq 2$

$$|A_0(z)| \leq \left(1+\sum_{i=1}^{k-1}|A_i(z)|\right)\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho+\varepsilon}\right\}. \tag{3.6}$$

According to assumption (1.2) and Lemma 2.2, for any $\mu>0$, there exists a set $E_1 \subset [0,1)$ with $\int_{E_1}\frac{dr}{1-r}=\infty$ such that for all z satisfying $|z|=r\in E_1$, we get

$$\frac{\sum_{i=1}^{k-1}|A_i(z)|+1}{|A_0(z)|}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}<1.$$

Then, for any $z\in\Delta$ satisfying $|z|=r\in E_1$, we have

$$|A_0(z)| > \left(\sum_{i=1}^{k-1}|A_i(z)|+1\right)\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}. \tag{3.7}$$

If $z\in\{z\in\Delta : |z|=r\in E_1\backslash E\}$ such that $\int_{E_1\backslash E}\frac{dr}{1-r}=\infty$ we obtain (3.7) contradicts with (3.5) when $p=q=1$

$$\exp\left\{\lambda\left(\frac{1}{1-r}\right)^{\mu}\right\}<\left(\frac{1}{1-r}\right)^{k(\rho+2+\varepsilon)}$$

and if $z\in\{z\in\Delta : |z|=r\in E_1\backslash E\}$ such that $\int_{E_1\backslash E}\frac{dr}{1-r}=\infty$ we obtain (3.7) contradicts with (3.6) when $p\geq q\geq 2$

$$\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}<\exp_{p-1}\left\{\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\rho+\varepsilon}\right\}.$$

Hence

$$\rho_{[p,q]}(f)=\infty.$$

By Lemma 2.1, for any given $\varepsilon>0$, there exists $E\subset[0,1)$ with $\int_E\frac{dr}{1-r}<\infty$ such that for all z satisfying $|z|=r\notin E$, we have

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\leq \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-r};T(s(r),f)\right\}\right)^{j}\\ &\leq \left(\frac{1}{1-r}\right)^{j(2+\varepsilon)}[T(s(r),f)]^{j}, \end{aligned} \tag{3.8}$$

where $s(r) = 1 - d(1-r)$, $d \in (0,1)$. Combining (3.4) and (3.8), we obtain

$$|A_0(z)| \leq \left(1 + \sum_{i=1}^{k-1} |A_i(z)|\right) \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} [T(s(r), f)]^k \tag{3.9}$$

for all $z \in \Delta$ such that $|z| = r \in E_1 \backslash E$, and by combining (3.9) and (3.7), we get

$$\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-r} \right) \right)^{\mu} \right\} < \left(\frac{1}{1-r}\right)^{k(2+\varepsilon)} [T(s(r), f)]^k. \tag{3.10}$$

We put $s(r) = R$, so $1 - |z| = \frac{1}{d}(1-R)$ and $\int\limits_E \frac{dr}{1-r} < +\infty$, then we can write (3.10) in the following form

$$\exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{d}{1-R} \right) \right)^{\mu} \right\} \left(\frac{1-R}{d}\right)^{k(2+\varepsilon)} \leq [T(R, f)]^k, \tag{3.11}$$

where $R \in d(E_1 \backslash E) + 1 - d$. Obviously the set $\{d(E_1 \backslash E) + 1 - d\}$ is of infinite logarithmic measure. Thus, by (3.11) we obtain

$$\rho_{[p+1,q]}(f) = \limsup_{R \to 1^-} \frac{\log_{p+1}^+ T(R, f)}{\log_q \left(\frac{1}{1-R}\right)} \geq \mu.$$

$\square$

Proof of Theorem 1.15

Proof. Assume that f is a nontrivial meromorphic solution of (1.1) with finite $[p,q]$ −order $\rho_{[p,q]}(f) = \rho < \infty$. If $p = q = 1$, then according to Lemma 2.3, there exists a set $E_2 \subset [0,1)$ with $\int\limits_{E_2} \frac{dr}{1-r} < \infty$, such that for $z \in \Delta$ satisfying $|z| = r \notin E_2$, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O\left(\log\left(\frac{1}{1-r}\right)\right), \ j = 1, ..., k. \tag{3.12}$$

By the definition of $\lambda\left(\frac{1}{A_0}\right)$, for any given ε $\left(0 < 2\varepsilon < \mu - \lambda\left(\frac{1}{A_0}\right)\right)$ and sufficiently large r, we have

$$N(r, A_0) \leq \left(\frac{1}{1-r}\right)^{\lambda\left(\frac{1}{A_0}\right)+\varepsilon}. \tag{3.13}$$

From (3.3), we obtain

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(1). \tag{3.14}$$

By (3.12), (3.13) and (3.14), we get

$$\begin{aligned} T(r, A_0) &= m(r, A_0) + N(r, A_0) \\ &\leq \sum_{i=1}^{k-1} m(r, A_i) + N(r, A_0) + O\left(\log\left(\frac{1}{1-r}\right)\right) \end{aligned}$$

$$\leq \sum_{i=1}^{k-1} T(r, A_i) + \left(\frac{1}{1-r}\right)^{\lambda\left(\frac{1}{A_0}\right)+\varepsilon} + O\left(\log\left(\frac{1}{1-r}\right)\right), \ \forall |z| \notin E_2. \tag{3.15}$$

If $p \geq q \geq 2$, by Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $E_3 \subset [0, 1)$ with $\int_{E_3} \frac{dr}{1-r} < \infty$, such that for $z \in \Delta$ satisfying $|z| = r \notin E_3$, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right), \forall j = 1, ..., k. \tag{3.16}$$

By the definition of $\lambda_{[p,q]}\left(\frac{1}{A_0}\right)$, for any given ε $\left(0 < 2\varepsilon < \mu - \lambda_{[p,q]}\left(\frac{1}{A_0}\right)\right)$ and sufficiently large r, we have

$$N(r, A_0) \leq \exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_0}\right) + \varepsilon\right)\log_q \frac{1}{1-r}\right\}. \tag{3.17}$$

By (3.14), (3.16) and (3.17), for all $|z| = r \notin E_3$, we obtain

$$\begin{aligned} T(r, A_0) &= m(r, A_0) + N(r, A_0) \\ &\leq \sum_{i=1}^{k-1} T(r, A_i) + \exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_0}\right) + \varepsilon\right)\log_q \frac{1}{1-r}\right\} \\ &\quad + O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right). \end{aligned} \tag{3.18}$$

According to assumption (1.3) and Lemma 2.2, for any $\mu > 0$, there exists a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} = \infty$, such that for $z \in \Delta$ satisfying $|z| = r \in E_1$, we have

$$\frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} < 1$$

which implies

$$T(r, A_0) - \sum_{i=1}^{k-1} T(r, A_i) > \exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}, \ \forall |z| = r \in E_1. \tag{3.19}$$

If $|z| = r \in E_1 \backslash E_2$, then by $0 < 2\varepsilon < \mu - \lambda\left(\frac{1}{A_0}\right)$, we see that (3.19) contradicts (3.15) when $p = q = 1$

$$\lambda\left(\frac{1}{1-r}\right)^{\mu} < \left(\frac{1}{1-r}\right)^{\lambda\left(\frac{1}{A_0}\right)+\varepsilon} + O\left(\log\left(\frac{1}{1-r}\right)\right)$$

and if $|z| = r \in E_1 \backslash E_3$, then by $0 < 2\varepsilon < \mu - \lambda_{[p,q]}\left(\frac{1}{A_0}\right)$, we see that (3.19) contradicts (3.18) when $p \geq q \geq 2$

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} < \exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_0}\right) + \varepsilon\right)\log_q \frac{1}{1-r}\right\}$$

$$+O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\frac{1}{1-r}\right\}\right)$$

$$=\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\lambda_{[p,q]}\left(\frac{1}{A_0}\right)+\varepsilon}\right\}$$

$$+O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q\frac{1}{1-r}\right\}\right).$$

Therefore, $\rho_{[p,q]}(f)=\infty$.
After that, from Lemma 2.3 there exists a set $E_2\subset[0,1)$ with $\int\limits_{E_2}\frac{dr}{1-r}<\infty$, such that for $z\in\Delta$ satisfying $|z|=r\notin E_2$, we have

$$m\left(r,\frac{f^{(j)}}{f}\right)\leq O\left(\log^+T(r,f)+\log\left(\frac{1}{1-r}\right)\right),\ j=1,...,k. \tag{3.20}$$

From (3.14), (3.17) and (3.20), we get

$$T(r,A_0)=m(r,A_0)+N(r,A_0)$$

$$\leq\sum_{i=1}^{k-1}m(r,A_i)+N(r,A_0)+O\left(\log^+T(r,f)+\log\left(\frac{1}{1-r}\right)\right)$$

$$\leq\sum_{i=1}^{k-1}T(r,A_i)+\exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_0}\right)+\varepsilon\right)\log_q\frac{1}{1-r}\right\}$$

$$+O\left(\log^+T(r,f)+\log\left(\frac{1}{1-r}\right)\right),\ \forall\,|z|\notin E_2. \tag{3.21}$$

If $|z|=r\in E_1\backslash E_2$ with $\int\limits_{E_1\backslash E_2}\frac{dr}{1-r}=\infty$, then combining (3.19) and (3.21), we get

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\leq\exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_0}\right)+\varepsilon\right)\log_q\frac{1}{1-r}\right\}$$

$$+O\left(\log^+T(r,f)+\log\left(\frac{1}{1-r}\right)\right). \tag{3.22}$$

Since ε satisfies $0<2\varepsilon<\mu-\lambda_{[p,q]}\left(\frac{1}{A_0}\right)$, then by (3.22), we have

$$(1-o(1))\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\leq O\left(\log^+T(r,f)+\log\left(\frac{1}{1-r}\right)\right),$$

therefore

$$\rho_{[p+1,q]}(f)=\limsup_{r\to1^-}\frac{\log^+_{p+1}T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}\geq\mu.$$

□

Proof of Theorem 1.19

Proof. Assume that f is a nontrivial meromorphic solution of (1.1) with finite $[p,q]$ −order $\rho_{[p,q]}(f)=\rho<\infty$. If $p=q=1$, then by Lemma 2.1, for any given $\varepsilon>0$, there exists a set $E\subset[0,1)$ with $\int\limits_{E}\frac{dr}{1-r}<\infty$, such that if $s+1\leq j\leq k$ and $|z|=r\notin E$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right|\leq O\left(\left(\frac{1}{1-r}\right)^{(j-s)(\rho+2+\varepsilon)}\right)$$

which implies

$$m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)=\frac{1}{2\pi}\int\limits_{0}^{2\pi}\log^{+}\left|\frac{f^{(j)}(re^{i\varphi})}{f^{(s)}(re^{i\varphi})}\right|d\varphi$$

$$\leq O\left(\log^{+}\left(\frac{1}{1-r}\right)\right),\ r\to1^{-}. \tag{3.23}$$

If $0\leq j\leq s-1$, then by using the first fundamental theorem of Nevanlinna, we have

$$T\left(r,\frac{f^{(j)}}{f^{(s)}}\right)=T\left(r,\frac{f^{(s)}}{f^{(j)}}\right)+O\left(1\right)=m\left(r,\frac{f^{(s)}}{f^{(j)}}\right)+N\left(r,\frac{f^{(s)}}{f^{(j)}}\right)+O\left(1\right).$$

Because $f^{(j)}$ has just finite many zeros, then according to the definition of the counting function, we obtain

$$N\left(r,\frac{f^{(s)}}{f^{(j)}}\right)=O(1)$$

so

$$m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)\leq T\left(r,\frac{f^{(j)}}{f^{(s)}}\right)\leq m\left(r,\frac{f^{(s)}}{f^{(j)}}\right)+O(1). \tag{3.24}$$

By the definition of $\lambda\left(\frac{1}{A_s}\right)$, for any given ε $\left(0<2\varepsilon<\mu-\lambda\left(\frac{1}{A_s}\right)\right)$ and sufficiently large r, we have

$$N(r,A_s)\leq\left(\frac{1}{1-r}\right)^{\lambda\left(\frac{1}{A_s}\right)+\varepsilon}. \tag{3.25}$$

From (1.1), we can write

$$-A_s\left(z\right)=\frac{f^{(k)}\left(z\right)}{f^{(s)}\left(z\right)}+A_{k-1}\left(z\right)\frac{f^{(k-1)}\left(z\right)}{f^{(s)}\left(z\right)}+\cdots+A_{s+1}\left(z\right)\frac{f^{(s+1)}\left(z\right)}{f^{(s)}\left(z\right)}$$

$$+A_{s-1}\left(z\right)\frac{f^{(s-1)}\left(z\right)}{f^{(s)}\left(z\right)}+\cdots+A_0\left(z\right)\frac{f'\left(z\right)}{f^{(s)}\left(z\right)}$$

which implies

$$m(r,A_s)\leq\sum_{i\neq s}m(r,A_i)+\sum_{s+1\leq j\leq k}m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)+\sum_{0\leq j\leq s-1}m\left(r,\frac{f^{(j)}}{f^{(s)}}\right)+O(1). \tag{3.26}$$

Combining (3.23), (3.24), (3.25) and (3.26), we obtain

$$T(r,A_s)=m(r,A_s)+N(r,A_s)$$

$$\leq \sum_{i\neq s} m(r,A_i) + N(r,A_s) + O\left(\log^+\left(\frac{1}{1-r}\right)\right)$$

$$\leq \sum_{i\neq s} T(r,A_i) + \left(\frac{1}{1-r}\right)^{\lambda\left(\frac{1}{A_s}\right)+\varepsilon} + O\left(\log^+\left(\frac{1}{1-r}\right)\right). \tag{3.27}$$

If $p \geq q \geq 2$, applying Lemma 2.4, there exists a set $E_3 \subset [0,1)$, with $\int_{E_3} \frac{dr}{1-r} < \infty$, such that $|z| = r \notin E_3$, we have (3.16) holds. By the definition of $\lambda_{[p,q]}\left(\frac{1}{A_s}\right)$, for any given ε $\left(0 < 2\varepsilon < \mu - \lambda_{[p,q]}\left(\frac{1}{A_s}\right)\right)$ and sufficiently large r, we have

$$N(r,A_s) \leq \exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_s}\right) + \varepsilon\right)\log_q \frac{1}{1-r}\right\}. \tag{3.28}$$

Since f just has finite many zeros, then when $s+1 \leq j \leq k$, by the definition of the counting function, we get

$$N\left(r, \frac{f^{(j)}}{f}\right) = O(1). \tag{3.29}$$

So, by using (3.16) and (3.29), we obtain

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leq m\left(r, \frac{f^{(j)}}{f}\right) + m\left(r, \frac{f}{f^{(s)}}\right)$$

$$\leq O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right) + T\left(r, \frac{f}{f^{(s)}}\right)$$

$$= O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right) + T\left(r, \frac{f^{(s)}}{f}\right) + O(1)$$

$$= O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right) + m\left(r, \frac{f^{(s)}}{f}\right) + N\left(r, \frac{f^{(s)}}{f}\right) + O(1)$$

$$= O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right). \tag{3.30}$$

Combining (3.24) when $0 \leq j \leq s-1$, (3.26), (3.28) and (3.30), for all $|z| = r \notin E_3$, we obtain

$$T(r,A_s) = m(r,A_s) + N(r,A_s)$$

$$\leq \sum_{i\neq s} m(r,A_i) + N(r,A_s) + O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right) \leq \sum_{i\neq s} T(r,A_i)$$

$$+ \exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_s}\right) + \varepsilon\right)\log_q \frac{1}{1-r}\right\} + O\left(\exp_{p-1}\left\{(\rho+\varepsilon)\log_q \frac{1}{1-r}\right\}\right). \tag{3.31}$$

From (1.4) and Lemma 2.2, for any $\mu > 0$, there exists a set $E_1 \subset [0,1)$, with $\int\limits_{E_1} \frac{dr}{1-r} = \infty$, such that for $|z| = r \in E_1$, we have

$$\frac{\prod\limits_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-r} \right) \right)^{\mu} \right\} < 1$$

which implies

$$T(r,A_s) - \sum_{i\neq s} T(r,A_i) > \exp_{p-1} \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-r} \right) \right)^{\mu} \right\}. \tag{3.32}$$

If $|z| = r \in E_1 \backslash E$, then by $0 < 2\varepsilon < \mu - \lambda\left(\frac{1}{A_s}\right)$, we see that (3.27) contradicts with (3.32) when $p = q = 1$

$$\lambda \left(\frac{1}{1-r} \right)^{\mu} < \left(\frac{1}{1-r} \right)^{\lambda\left(\frac{1}{A_s}\right)+\varepsilon} + O\left(\log^+ \left(\frac{1}{1-r} \right) \right)$$

and if $|z| = r \in E_1 \backslash E_3$, then by $0 < 2\varepsilon < \mu - \lambda_{[p,q]}\left(\frac{1}{A_s}\right)$, we see that (3.31) contradicts with (3.32) when $p \geq q \geq 2$

$$\exp_{p-1} \left\{ \lambda \left(\log_{q-1} \left(\frac{1}{1-r} \right) \right)^{\mu} \right\} < \exp_p \left\{ \left(\lambda_{[p,q]} \left(\frac{1}{A_s} \right) + \varepsilon \right) \log_q \frac{1}{1-r} \right\}$$
$$+ O\left(\exp_{p-1} \left\{ (\rho + \varepsilon) \log_q \frac{1}{1-r} \right\} \right).$$

Hence, $\rho_{[p,q]}(f) = \infty$.

Next from Lemma 2.1, for any given $\varepsilon > 0$, there exists a set $E \subset [0,1)$ with $\int\limits_{E} \frac{dr}{1-r} < \infty$, such that if $s+1 \leq j \leq k$ and $|z| = r \notin E$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \leq \left(\left(\frac{1}{1-r} \right)^{2+\varepsilon} \max \left\{ \log \frac{1}{1-r}; T(s(r), f) \right\} \right)^{j-s},$$

where $s(r) = 1 - d(1-r)$, $d \in (0,1)$. We have $\lim\limits_{r\to 1^-} \frac{s(r)}{r} = 1$, so $\log \frac{1}{1-r} > 1$ when $r \to 1^-$ and $T(s(r), f) = T(r,f)$. Therefore

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(j)}(re^{i\varphi})}{f^{(s)}(re^{i\varphi})} \right| d\varphi$$

$$\leq \log^+ \left(\left(\frac{1}{1-r} \right)^{2+\varepsilon} \left\{ \log \frac{1}{1-r} + T(s(r), f) \right\} \right)^{j-s}$$

$$\leq O\left(\log^+ T(r,f) + \log^+ \left(\frac{1}{1-r} \right) \right), \quad r \to 1^-. \tag{3.33}$$

Combining (3.24) when $0 \leq j \leq s-1$, (3.26), (3.28) and (3.33), we get for $r \in E_1 \backslash E$

$$T(r,A_s) = m(r,A_s) + N(r,A_s)$$

$$\leq \sum_{i\neq s} m(r,A_i)+N(r,A_s)+O\left(\log^+ T(r,f)+\log^+\left(\frac{1}{1-r}\right)\right)$$
$$\leq \sum_{i\neq s} T(r,A_i)+\exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_s}\right)+\varepsilon\right)\log_q \frac{1}{1-r}\right\}$$
$$+O\left(\log^+ T(r,f)+\log^+\left(\frac{1}{1-r}\right)\right), \tag{3.34}$$

where the set $E_1\backslash E$ is of infinite logarithmic measure. By (3.32) and (3.34), we have

$$\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\leq \exp_p\left\{\left(\lambda_{[p,q]}\left(\frac{1}{A_s}\right)+\varepsilon\right)\log_q \frac{1}{1-r}\right\}$$
$$+O\left(\log^+ T(r,f)+\log^+\left(\frac{1}{1-r}\right)\right)$$

and since $0<2\varepsilon<\mu-\lambda_{[p,q]}\left(\frac{1}{A_s}\right)$, we obtain

$$(1-o(1))\exp_{p-1}\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\leq O\left(\log^+ T(r,f)+\log^+\left(\frac{1}{1-r}\right)\right). \tag{3.35}$$

Thus, from (3.35) we get

$$\rho_{[p+1,q]}(f)=\limsup_{r\to 1^-}\frac{\log^+_{p+1} T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}\geq \mu.$$

□

Proof of Theorem 1.23

Proof. Assume that f is a nontrivial meromorphic solution of (1.1) with $\rho_{[p,q]}(f)=\infty$ and $\rho_{[p+1,q]}(f)=\rho<\infty$. Set $g(z)=f(z)-\varphi(z)$. Then, by Lemma 2.5, $g(z)$ is a meromorphic function with $\rho_{[p,q]}(g)=\infty$ and $\rho_{[p+1,q]}(g)=\rho<\infty$. Substituting $f=g+\varphi$ into (1.1), we get

$$g^{(k)}+A_{k-1}(z)\,g^{(k-1)}+\cdots+A_1(z)\,g'+A_0(z)\,g=H, \tag{3.36}$$

where

$$H=-\left[\varphi^{(k)}+A_{k-1}(z)\,\varphi^{(k-1)}+\cdots+A_1(z)\,\varphi'+A_0(z)\,\varphi\right].$$

We prove that $H\not\equiv 0$. In fact, if $H\equiv 0$, then

$$\varphi^{(k)}+A_{k-1}(z)\,\varphi^{(k-1)}+\cdots+A_1(z)\,\varphi'+A_0(z)\,\varphi=0.$$

Hence $\rho_{[p,q]}(\varphi)=+\infty$, which is a contradiction. Therefore $H\not\equiv 0$. We know that the functions A_j $(j=0,1,...,k-1)$ and H satisfy

$$\max\left\{\rho_{[p,q]}(A_j)\quad(j=0,...,k-1),\rho_{[p,q]}(H)\right\}<\rho_{[p,q]}(f)=\infty.$$

By Lemma 2.6 and (3.36), we have

$$\overline{\lambda}_{[p,q]}(g)=\lambda_{[p,q]}(g)=\rho_{[p,q]}(g)=\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g)=\lambda_{[p+1,q]}(g)=\rho_{[p+1,q]}(f)=\rho.$$

Therefore

$$\overline{\lambda}_{[p,q]}\left(f-\varphi\right)=\lambda_{[p,q]}\left(f-\varphi\right)=\rho_{[p,q]}\left(f\right)=\infty$$

and

$$\overline{\lambda}_{[p+1,q]}\left(f-\varphi\right)=\lambda_{[p+1,q]}\left(f-\varphi\right)=\rho_{[p+1,q]}\left(f\right)=\rho.$$

which completes the proof. $\square$

4. Examples

The following example shows that the Theorem 1.12 is sharp.

Example 4.1. Consider the following equation

$$f''+A_1(z)f'+A_0(z)f=0, \tag{4.1}$$

where $A_0(z)=\frac{7}{z}\exp\left\{2\exp_6\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$ and $A_1(z)=\frac{1}{z}\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$. Then, for $w_0=1\in\partial\Delta$ and a curve $\gamma=\{z\in\Delta:\arg z=0\}\subset\Delta$ tending to $w_0=1$ and two constants $\mu=7$ and $\lambda=1$ and for $p=q=7$, we have

$$\lim_{\substack{|z|\to1^-\\ z\in\gamma}}\frac{|A_1(z)|+1}{|A_0(z)|}\exp_p\left\{\lambda\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^\mu\right\}$$

$$=\lim_{\substack{|z|\to1^-\\ z\in\gamma}}\frac{\left|\frac{1}{z}\right|\left|\exp_7\left\{\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}\right|+1}{7\left|\frac{1}{z}\right|\left|\exp\left\{2\exp_6\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}\right|}\exp_7\left\{\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}$$

$$=\lim_{r\to1^-}\frac{\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+r}{7\exp\left\{2\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}}\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}$$

$$=\lim_{r\to1^-}\frac{1}{7}\left(\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+r\right)\times$$
$$\exp\left\{\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7-2\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}$$

$$=\lim_{r\to1^-}\frac{1}{7}\left(\exp_7\left\{\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}+r\right)\exp\left\{-\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}$$

$$=\lim_{r\to1^-}\frac{1}{7}\left(\exp\left\{\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7-\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}\right.$$
$$\left.+r\exp\left\{-\exp_6\left(\log_6\left(\frac{1}{1-r}\right)\right)^7\right\}\right)=\frac{1}{7}<1.$$

Thus, by Theorem 1.12, every nontrivial meromorphic solution f of (4.1) satisfies $\rho_{[7,7]}(f)=\infty$ and $\rho_{[8,7]}(f)\geq7$.

The following example shows that the Corollary 1.13 is sharp.

Example 4.2. Consider the following equation

$$f'' + A_1(z)f' + A_0(z)f = 0, \tag{4.2}$$

where

$$A_0(z) = K_0(z)\exp_7\left\{2\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}$$

and

$$A_1(z) = K_1(z)\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\},$$

such that K_0, K_1 are meromorphic functions in Δ and analytic at the point $w_0 = 1$ satisfying

$$|K_1(z)| \leq 1, \quad |K_0(z)| \geq 1.$$

Then, for $w_0 = 1 \in \partial\Delta$ and a curve $\gamma = \{z \in \Delta : \arg z = 0\} \subset \Delta$ tending to $w_0 = 1$ and two constants $\mu = 7$ and $\lambda = 1$ and for $p = q = 7$, we have

$$|A_0(z)| = |K_0(z)|\left|\exp_7\left\{2\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}\right| \geq \exp_7\left\{2\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}$$

and

$$|A_1(z)| = |K_1(z)|\left|\exp_7\left\{\left(\log_6\left(\frac{1}{1-z}\right)\right)^7\right\}\right| \leq \exp_7\left\{\left(\log_6\left(\frac{1}{1-|z|}\right)\right)^7\right\}.$$

Hence, by Corollary 1.13 (when $\alpha = 2$ and $\beta = 1$), every nontrivial meromorphic solution f of (4.2) satisfies $\rho_{[7,7]}(f) = \infty$ and $\rho_{[8,7]}(f) \geq 7$.

References

1. B. Belaïdi, *Growth of solutions to linear equations with analytic coefficients of [p, q]-order in the unit disc*, Electron. J. Differential Equations. **1** (2011), 1-11.
2. B. Belaïdi, *Growth and oscillation theory of [p, q]-order analytic solutions of linear equations in the unit disc*, J. Math. Anal. **3** (2012), 1-11.
3. B. Belaïdi, *On the [p, q]-order of analytic solutions of linear equations in the unit disc*, Novi Sad J. Math. **42** (2012), 117-129.
4. T. B. Cao and H. X. Yi, *The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc*, J. Math. Anal. Appl. **319** (2006), 278-294.
5. T. B. Cao, *The growth, oscillation and fixed points of solutions of complex linear differential equations in the unit*, J. Math. Anal. Appl. **352** (2009), 739-748.
6. Y. Chen and G. Deng, *Fast growth and fixed points of solutions of higher-order linear differential equations in the unit disc*, AIMS Math. **6** (2021), 10833-10845.
7. I. Chyzhykov, G. Gundersen and J. Heittokangas, *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc. (3). **86** (2003), 735-754.

8. S. Hamouda, *Iterated order of solutions of linear differential equations in the unit disc*, Comput. Methods Funct. Theory. **13** (2013), 545–555.
9. W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford. (1964).
10. J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. **122** (2000), 1-54.
11. J. Heittokangas, R. Korhonen and J. Rättyä, *Fast growing solutions of linear differential equations in the unit disc*, Results Math. **49** (2006), 265-278.
12. H. Hu and X. M. Zheng, *Growth of solutions of linear differential equations with analytic coefficients of [p,q]-order in the unit disc*, Electron. J. Differential Equations. **2014** (2014), 1-12.
13. I. Laine, *Complex differential equations. Handbook of differential equations: ordinary differential equations*, Vol. IV, 269–363, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.
14. Z. Latreuch and B. Belaïdi, *Linear differential equations with analytic coefficients of [p,q]-order in the unit disc*, Sarajevo J. Math. **9** (2013), 71-84.
15. H. Qin, J. Long and M. Li, *On [p, q]-order of growth of solutions of linear differential equations in the unit disc*, AIMS Math. **6** (2021), 12878-12893.
16. M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea, New York, (1975), reprint of the 1959 edition.
17. J. Tu and H. X. Huang, *Complex oscillation of linear differential equations with analytic coefficients of [p,q]-order in the unit disc*, Comput. Methods Funct. Theory. **15** (2015), 225-246.
18. J. Tu and Z. X. Xuan, *Complex linear differential equations with certain analytic coefficients of [p, q]-order in the unit disc*, Adv. Difference Equ. Theory. **2014** (2014), 1-12.

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