

LOGARITHMIC AND INVERSE LOGARITHMIC COEFFICIENTS OF CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. The objective of this article is to compute upper bounds for the logarithmic coefficients γ_n and inverse logarithmic coefficients Γ_n , $n=1,2,3$ of certain subclasses of analytic univalent functions defined in the unit disk.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{S}$ denote the class of normalised analytic univalent functions of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n,$$

in the open unit disk $\mathbb{U} = \{\xi : |\xi| < 1\}$. The logarithmic coefficients of f are defined by

$$\log \frac{f(\xi)}{\xi} = 2 \sum_{n=1}^{\infty} \gamma_n \xi^n. \quad (1.1)$$

Exact upper bounds for γ_n of few subclasses of $\mathcal{S}$ have been established [4, 5, 7]. Moreover we know that for $f \in \mathcal{S}$ the expected inequality $|\gamma_n| \leq \frac{1}{n}$ is false even in order of magnitude.

Therefore considering the definition of different subclasses defined by various authors [14, 12, 6]. Ye [15], Murat Caglar [11] estimated the general logarithmic coefficient for close to convex functions. Al-Amiri and Fernando [3] introduced the class $S_c(b)$ of close-to-convex functions of complex order b .

The initial logarithmic coefficients $\gamma_1, \gamma_2, \gamma_3$ for $f \in \mathcal{S}$ can be obtained by differentiating (1.1)

$$\gamma_1 = \frac{1}{2}a_2, \quad (1.2)$$

$$\gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2), \quad (1.3)$$

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$$\gamma_3 = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3). \quad (1.4)$$

We now recall the concept of subordination of two analytic functions. A function $f \in \mathcal{S}$ is said to be subordinate to $g \in \mathcal{S}$ denoted by $f \prec g$ if there exists a Schwarz function ϕ that is analytic in $\mathbb{U}$ with $\phi(0) = 0$ and $|\phi(\xi)| < 1$ such that $f(\xi) = g(\phi(\xi))$ for $\xi \in \mathbb{U}$.

In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Using subordination, Ma and Minda [10] introduced two different subclasses $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ of starlike and convex functions, in which either of the quantity $\frac{\xi f'(\xi)}{f(\xi)}$

or $1 + \frac{\xi f''(\xi)}{f'(\xi)}$ is subordinate to a more general superordinate function. They considered an analytic univalent function ϕ , $\phi(0) = 1$ and $\phi'(0) > 0$ with positive real part in $\mathbb{U}$, $\phi(\mathbb{U})$ is symmetric with respect to the real axis and starlike. The series expansion of ϕ is $\phi(\xi) = 1 + \sum_{k=1}^{\infty} c_k \xi^k$.

Symbolically

$$\begin{aligned} \mathcal{S}^*(\phi) &= \left\{ f \in \mathcal{S} : \frac{\xi f'(\xi)}{f(\xi)} \prec \phi(\xi) \right\} \\ \mathcal{C}(\phi) &= \left\{ f \in \mathcal{S} : 1 + \frac{\xi f''(\xi)}{f'(\xi)} \prec \phi(\xi) \right\}. \end{aligned}$$

In Geometric function theory, some special subclasses of the class $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ play an important role because of their geometric properties.

Definition 1.1. [12] Let $f \in \mathcal{S}$ and $0 < \alpha \leq 1$ we say that $f \in R(\alpha)$ if

$$\frac{f'(\xi) - 1}{f'(\xi) + 1} \prec \alpha \xi, \quad \xi \in \mathbb{U}. \quad (1.5)$$

Definition 1.2. [1] Let $f \in \mathcal{S}$ and $0 \leq \lambda < 1$, $0 \leq \beta < 1$ we say that $f \in M(\lambda, \beta)$ if

$$\operatorname{Re} \left\{ \frac{\xi f'(\xi)}{(1 - \lambda)f(\xi) + \lambda \xi f'(\xi)} \right\} > \beta, \quad \xi \in \mathbb{U}. \quad (1.6)$$

In this article we estimate the upper bounds for the logarithmic coefficients γ_n , $n=1,2,3$ of the functions in $R(\alpha)$ and $M(\lambda, \beta)$ and its corresponding inverse logarithmic coefficients Γ_n , $n=1,2,3$.

2. MAIN RESULTS

2.1. Logarithmic coefficients of $R(\alpha)$ and $M(\lambda, \beta)$.

Theorem 2.1. *The logarithmic coefficients γ_i , $i = 1, 2, 3$ of $f \in R(\alpha)$ satisfy the inequalities:*

$$\begin{aligned} |\gamma_1| &\leq \frac{\alpha}{2} \\ |\gamma_2| &\leq \frac{2\alpha}{3} + \frac{\alpha^2}{12} \end{aligned}$$

$$|\gamma_3| \leq \alpha + \frac{\alpha^2}{3} + \frac{\alpha^3}{12}.$$

Proof. Since $f \in R(\alpha)$ there exists a Schwarz function ϕ such that

$$\frac{f'(\xi) - 1}{f'(\xi) + 1} = \alpha\phi(\xi)$$

Equating the coefficients,

$$a_2 = c_1\alpha \tag{2.1}$$

$$a_3 = \frac{2\alpha}{3}(c_2 + c_1^2\alpha) \tag{2.2}$$

$$a_4 = \frac{\alpha}{2}(c_3 + 2c_1c_2\alpha + c_1^3\alpha^2). \tag{2.3}$$

Substituting the above in (1.2), (1.3), (1.4) we get

$$\gamma_1 = \frac{c_1\alpha}{2} \tag{2.4}$$

$$\gamma_2 = \frac{c_2\alpha}{3} + \frac{c_1^2\alpha^2}{12} \tag{2.5}$$

$$\gamma_3 = \frac{c_3\alpha}{4} + \frac{c_1c_2\alpha^2}{6} + \frac{c_1^3\alpha^3}{12}. \tag{2.6}$$

We know that

$$\phi(\xi) = \frac{p(\xi) - 1}{p(\xi) + 1}$$

where $Re(p(\xi)) > 0$ and $p(\xi) = 1 + \sum_{k=1}^{\infty} p_k \xi^k$. Since $|p_i| \leq 2$, (See[6]) we get $|c_1| \leq 1$, $|c_2| \leq 2$, and $|c_3| \leq 4$.

Using the bounds for $|c_i|$ in (2.4), (2.5), (2.6) we obtain

$$|\gamma_1| \leq \frac{\alpha}{2}$$

$$|\gamma_2| \leq \frac{2\alpha}{3} + \frac{\alpha^2}{12}$$

$$|\gamma_3| \leq \alpha + \frac{\alpha^2}{3} + \frac{\alpha^3}{12}.$$

□

Theorem 2.2. *The logarithmic coefficients $\gamma_i, i = 1, 2, 3$ of $f \in M(\lambda, \beta)$ satisfy the inequalities:*

$$|\gamma_1| \leq \frac{|1 - \beta|}{|1 - \lambda|}$$

$$|\gamma_2| \leq \frac{|1 - \beta|}{|2(1 - \lambda)^2|} |\lambda(1 - 2\beta) + 1|$$

$$|\gamma_3| \leq \frac{|1 - \beta|}{|3(1 - \lambda)^3|} \left| (1 - \beta) [16\lambda(1 - \beta) - 4\lambda^2(1 + 2\beta) + 4\lambda(1 - 2\lambda^2) + (1 - \lambda)(2\lambda^2 + 10\lambda + 3)] \right. \\ \left. + (1 - \lambda)[3\lambda^2 - 9\lambda + 1] \right|.$$

Proof. As $f \in M(\lambda, \beta)$

$$\frac{\xi f'(\xi)}{(1-\lambda)f(\xi) + \lambda \xi f'(\xi)} = \beta + (1-\beta)h(\xi)$$

where $h(\xi)$ is a positive function with $h(0) = 1$ and $Re(h(\xi)) > 0$. Equating the coefficients we get

$$a_2 = \frac{h_1(1-\beta)}{(1-\lambda)} \quad (2.7)$$

$$a_3 = \frac{1-\beta}{2(1-\lambda)} \left[h_1^2 \left((1-\beta) + \frac{2\lambda}{1-\lambda} \right) + h_2\lambda + h_3(1-\lambda) \right] \quad (2.8)$$

$$a_4 = \frac{1-\beta}{3(1-\lambda)} \left[(1-\beta)(1+2\lambda)h_1 \left(\frac{h_1^2(1-\beta)}{2(1-\lambda)} + \frac{\lambda h_1^2}{1-\lambda} + \frac{h_3}{2} \right) \right. \\ \left. + (1-\beta)h_1 \left(\frac{\lambda h_2(1+2\lambda)}{2(1-\lambda)} + \frac{h_2(1+\lambda)}{1-\lambda} \right) + h_3 \right] \quad (2.9)$$

Now substituting the above in (1.2), (1.3), (1.4) and since $|h_i| \leq 2$, we get

$$|\gamma_1| \leq \frac{|1-\beta|}{|1-\lambda|}$$

$$|\gamma_2| \leq \frac{|1-\beta|}{|2(1-\lambda)^2|} |\lambda(1-2\beta) + 1|$$

$$|\gamma_3| \leq \frac{|1-\beta|}{|3(1-\lambda)^3|} \left| (1-\beta) [16\lambda(1-\beta) - 4\lambda^2(1+2\beta) + 4\lambda(1-2\lambda^2) + (1-\lambda)(2\lambda^2 + 10\lambda + 3)] \right. \\ \left. + (1-\lambda)[3\lambda^2 - 9\lambda + 1] \right|.$$

□

2.2. Inverse Logarithmic coefficients of $R(\alpha)$ and $M(\lambda, \beta)$. Let F be the inverse function of $f \in S$ defined in the neighbourhood of the origin with the Taylors series expansion

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad (2.10)$$

where we may choose $|w| < \frac{1}{4}$. The logarithmic coefficients Γ_n , $n \in \mathbb{N}$ of F are defined by the equation

$$\log \frac{F(w)}{w} = 2 \sum_{n=1}^{\infty} \Gamma_n w^n. \quad (2.11)$$

We observe that $f(F(w)) = w$, $F(0) = 0 = f(0)$ and $F'(0) = 1 = f'(0)$ where $\xi = F(w)$. Differentiating we find $f'(\xi)F'(w) = 1$.

On differentiating further

$$f''(\xi)(F'(w))^2 + f'(\xi)F''(w) = 0$$

$$f'''(\xi)(F'(w))^3 + 3f''(\xi)F'(w)F''(w) + f'(\xi)F'''(w) = 0$$

$$f^{iv}(\xi)(F'(w))^4 + 6f'''(\xi)(F'(w))^2F''(w) + f''(\xi)(3(F''(w))^2 + 3F'(\xi)F'''(w) + F'''(w)) + f'(\xi)F^{iv}(w) = 0$$

Setting $\xi = 0, w = 0$ we obtain that

$$A_2 = -a_2 \quad (2.12)$$

$$A_3 = -a_3 + 2a_2^2 \quad (2.13)$$

$$A_4 = -a_4 + 5a_2a_3 - 5a_2^3. \quad (2.14)$$

Comparing the coefficients we get

$$2\Gamma_1(F) = A_2,$$

$$2\Gamma_2(F) = A_3 - \frac{1}{2}A_2^2,$$

$$2\Gamma_3(F) = A_4 - A_2A_3 + \frac{1}{3}A_2^3.$$

Using (2.12), (2.13), (2.14) above we get (See[13])

$$2\Gamma_1(F) = -a_2, \quad (2.15)$$

$$4\Gamma_2(F) = -2a_3 - 3a_2^2, \quad (2.16)$$

$$6\Gamma_3(F) = -3a_4 + 12a_2a_3 - 10a_2^3. \quad (2.17)$$

Theorem 2.3. *If F is the inverse of $f \in R(\alpha)$, then*

$$|\Gamma_1(F)| \leq \frac{\alpha}{2},$$

$$|\Gamma_2(F)| \leq \frac{13\alpha}{12},$$

$$|\Gamma_3(F)| \leq 2\alpha + \frac{5\alpha^2}{3} + \frac{7\alpha^3}{12}.$$

Proof. Substituting $|c_1| \leq 1$, $|c_2| \leq 2$ and $|c_3| \leq 4$ in (2.1), (2.2), (2.3) we get

$$|a_2| \leq \alpha$$

$$|a_3| \leq \frac{2\alpha}{3}(2 + \alpha)$$

$$|a_4| \leq \frac{\alpha}{2}(4 + 4\alpha + \alpha^2).$$

Using the above in (2.15), (2.16), (2.17) we get

$$|\Gamma_1(F)| \leq \frac{\alpha}{2},$$

$$|\Gamma_2(F)| \leq \frac{13\alpha}{12},$$

$$|\Gamma_3(F)| \leq 2\alpha + \frac{5\alpha^2}{3} + \frac{7\alpha^3}{12}.$$

□

Theorem 2.4. *If F is the inverse of $f \in M(\lambda, \beta)$, then*

$$|\Gamma_1(F)| \leq \frac{|(1-\beta)|}{|(1-\lambda)|},$$

$$|\Gamma_2(F)| \leq \frac{|1-\beta|}{|4(1-\lambda)|} [(1-\beta)(\lambda^2 - 2\lambda + 7) + (1-\lambda)(6\lambda + 2)],$$

$$|\Gamma_3(F)| \leq \frac{|(1-\beta)^2|}{|(1-\lambda)|^2} \left[\frac{(1+2\lambda)}{3} (3(1+\lambda) - 2\beta) + \frac{2(\lambda^2 + \lambda + 1)}{3} + \frac{2(1-\lambda)}{3(1-\beta)} + 2(4(1-\beta)(1-\lambda) + 6\lambda + 2) + \frac{40(1-\beta)}{3(1-\lambda)} \right].$$

Proof. Since $|h_i| \leq 2$ equations (2.7), (2.8), (2.9) can be written as

$$|a_2| \leq \frac{|2(1-\beta)|}{|(1-\lambda)|},$$

$$|a_3| \leq \frac{|1-\beta|}{|2(1-\lambda)|} [4(1-\beta)(1-\lambda) + 6\lambda + 2],$$

$$|a_4| \leq \frac{|2(1-\beta)|}{|3(1-\lambda)|} \left[\frac{(1-\beta)(1+2\lambda)}{(1-\lambda)} (3(1+\lambda) - 2\beta) + \frac{2(1-\beta)}{(1-\lambda)} (\lambda^2 + \lambda + 1) + 2 \right].$$

Computing in lines similar to Theorem 2.3 we get the upper bounds for the inverse logarithmic coefficients of $M(\lambda, \beta)$.

□

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