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RECENT FIXED POINT THEOREMS ON $b_v(s)-$METRIC SPACE

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Abstract. In this paper we have established solution of recent open problems of H. Garai et al.[2] on $b_v(s)-$metric space and a theorem of Hardy Rogers type contraction condition in metric space that generalized many existing results.

1. Introduction and preliminaries

It has observed that the class of $b_v(s)$ metric spaces is larger than all other metric spaces. Many authors such as [3],[5], [7] have worked on $b-$metric space, rectangular $b-$metric space. Also many researchers have given generalizations and interesting results using various contraction such as Banach, Kannan, Reich etc. So it is natural to ask whether the fixed point results regarding the various contractive conditions can be proved in the $b_v(s)$ metric space or not.

In this paper we have proved the open question of H. Garai et al. [2] and have proved a theorem using Hardy Rogers type contraction condition on $b_v(s)$ metric space.

2. Definitions

Definition 2.1. [2] Let X be a non-empty set, $v \in \mathbb{N}$ and $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}$ is a said to be a $b_v(s)$ metric if for all distinct points $u_1, u_2, ..., u_v \in X$, all are different from x and y, the following conditions holds:
(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + ... + d(u_v, y)]$.

If d is a $b_v(s)$ metric on X, then the pair (X, d) is a called the $b_v(s)$ metric space.
Note: [2]
i) $b_1(1)$ metric space is the usual metric space.
ii) $b_1(s)$ metric space is $b-$metric space with coefficient s.
iii) $b_2(1)$ metric space is rectangular metric space.

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iv) $b_2(s)$ metric space is rectangular $b-$metric space with coefficient s.
v) $b_v(1)$metric space is $v-$generalized metric space.
Mitrović and Radenović[4] introduced the notions of Cauchy sequences and completeness in bv(s)-metric space.

Definition 2.2. [2] Let (X, d) be a $b_v(s)$-metric space, (x_n) be a sequence in X and $x \in X$.

(i) The sequence (x_n) is said to be a Cauchy sequence, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > N$ and for all $p \in N$.

(ii) The sequence (x_n) is said to be convergent to x, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > N$ and this fact is represented by $lim_{n\to\infty} x_n = x$.

(iii) (X, d) is said to be a complete $bv(s)$-metric space if every Cauchy sequence in X converges to some $x \in X$.

Theorem 2.3. [1] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$, and let for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon < d(x, y) < \epsilon + \delta \Rightarrow d(Tx, Ty) \leq \epsilon$$

for any $x, y \in X$. Then T has a unique fixed point z, and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to z.

Suzuki [6] introduced another additional weaker assumption with the completeness of (X, d) to assure fixed point of T in the following theorem.

Theorem 2.4. [6] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$ and let the following hold:

For any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(T^i x, T^j x) < \epsilon + \delta \Rightarrow d(T^{i+1}x; T^{j+1}x) \leq \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T has a unique fixed point z, and for any $x \in X$, the sequence of iterates $(\{T^n x\}$. converges to z.

After that H. Garai et al. [2] have proved theorem:

Theorem 2.5. [2] *Let (X, d) be a complete $b_v(1)$-metric space, and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) < d(x, y)$$

holds for all $x, y \in X$ with $x \neq y$. Further, assume that for any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(T^i x, T^j x) < \epsilon + \delta \Rightarrow d(T^{i+1}x, T_{j+1}x) \leq \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.

They have given an open problem as:
Open Question 4.11. Let (X, d) be a complete $b_v(s)$-metric space and let T be a self-mapping on X such that

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. If $s > 1$, then find out a weaker additional assumption on T which will ensure that T has a fixed point.

3. Main Results

The following theorem is the solution of said open problem.

Theorem 3.1. *Let (X, d) be a complete $b_v(s)$-metric space where $s > 1$, and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) < d(x, y)$$

holds for all $x, y \in X$ with $x \neq y$. Further, assume that for any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(T^i x, T^j x) < \epsilon + \delta \Rightarrow d(T^{i+1}x, T^{j+1}x) \leq \epsilon$$

for any $i, j \in \mathbb{N} \cup \{0\}$. Then T possesses a unique fixed point, and for any $x \in X$, the Picard's iterative sequence $(T^n x)$ converges to that fixed point.

Proof. Let us choose an initial approximation $x_0 \in X$. We construct a sequence $\{x_n\}$ in X, such That $x_n = T^n x_0$ for every natural number n. If $x_n = x_{n+1}$ for some natural number n, then x_n is the unique fixed point of T. So we assume that $x_n \neq x_{n+1}$ for every natural numbers n. So we asume that all the terms of the squence $\{x_n\}$ are distinct.

Next, we consider the sequence of real numbers $\{d_n\}$ where $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then

$d_{n+1} = d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) = d_n$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is a monotone decreasing sequence of non-negative real numbers and hence converges to $a \geq 0$ (say).
Therefore,

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_{x_n}) < d(x_{n-1}, x_n) = a,$$

which is a contradiction.
Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.1}$$

Again let, $d_n^* = d(x_n, x_{n+2}) = d(Tx_{n_1}, Tx_{n+1}) < d(x_{n-1}, x_{n+1})$.
Therefore, $d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1}) < d(x_{n-2}, x_n) < \ldots < d(x_0, x_2)$.
So again $\{d(x_n, x_{n+2})\}$ is a monotone decreasing sequence of real numbers.

Suppose it converges to to b.
Then

$b = \lim_{n\to\infty} d(x_n, x_{n+2}) = \lim_{n\to\infty} d(Tx_{n-1}, x_{n+1}) < d(x_{n-1}, x_{n+1}) = b$, which leads to a contradiction.
Therefore, $b = 0$. i.e.,

$$\lim_{n\to\infty} d_n^* = \lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{3.2}$$

Now we show that $\{x_n\}$ is a Cauchy sequence.

We prove it by Mathematicsal Induction on $j \in \mathbb{N}$

$$\text{i.e., } \lim_{n\to\infty} d(x_{n+2v}, x_{n+2v+j}) = 0 \tag{3.3}$$

holds for all $j \in \mathbb{N}$.
Clearly (3.3) holds for $j = 1, 2$.

Suppose it holds for $j = k$. i.e., $\lim_{n\to\infty} d(x_{n+2v}, x_{n+2v+k}) = 0$.
So, $\lim_{n\to\infty} d(x_{n+3v}, x_{n+3v+k}) = 0$.
Since

$$\begin{aligned}
&d(x_{n+2v}, x_{n+2v+k+1})\\
&\leq s[d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + \ldots + d(x_{n+2v+v}, x_{n+2v+v+1})\\
&+ d(x_{n+2v+v+1}, x_{n+2v+k+1})]\\
&= s[d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + \ldots + d(x_{n+2v+v}, x_{n+2v+v+1})]\\
&+ sd(x_{n+2v+k+1}, x_{n+3v+1})\\
&\leq s[d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + \ldots + d(x_{n+2v+v}, x_{n+2v+v+1})]\\
&+s^2[d(x_{n+2v+k+1}, x_{n+2v+k+2})+d(x_{n+2v+k+2}, x_{n+2v+k+3})+\ldots+d(x_{n+2v+k+v}, x_{n+2v+k+v+1})+\\
&d(x_{n+2v+j+v+1}, x_{n+3v+1})]\\
&= s[d_{n+2v}^* + d_{n+2v+2} + \ldots + d_{n+3v}]\\
&+ s^2[d_{n+2v+j+1} + d_{n+2v+j+2} + \ldots + d_{n+3v+j} + d(x_{n+3v+j+1}, x_{n+3v+1})].
\end{aligned}$$

Therefore,

$\lim_{n\to\infty} d(x_{n+2v}, x_{n+2v+k+1}) = 0$ [since each of $d_n, d_n^* \to 0$ as $n \to \infty$].

This shows that $\{x_n\}$ is a Cauchy sequence. Since the space is complete, there exists an element $x \in X$ such that $x_n \to x$ as $n \to \infty$. As T is contractive, T is continuous. Since $x_n = Tx_{n-1}$, the sequence converges to Tx. This shows that x is a fixed point of T.

To show the uniqueness of the fixed point let x' another fixed point of T.
Then

$d(x, x') = d(Tx, Tx') < d(x, x')$,
which is a contradiction. Therefore, $x = x'$.
Hence the theorem. ☐

Theorem 3.2. *Let (X, d) be a complete $b_v(s)-$metric space with coefficient $s > 1$ and $T : X \to X$ be a self map satisfying the relation*

$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)$
wehere $\alpha_i \geq 0$ and $\alpha_1 + s\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$. Then T have a unique fixed point in X.

Proof. Let $x_0 \in x$ be an initial approximation. We construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n, n \in \mathbb{N}$. Clearly if $x_{n+1} = x_n$ i.e., $Tx_n = x_n$, then x_n is a fixed point of T for all $n \in \mathbb{N}$.

Suppose $d_n = d(x_n, x_{n+1})$ and $d_n^* = d(x_n, x_{n+2})$.
Now $d_n = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$\leq \alpha_1 d(x_{n-1},x_n)+\alpha_2 d(x_{n-1},Tx_{n-1})+\alpha_3 d(x_n,Tx_n)+\alpha_4 d(x_{n-1},Tx_n)+\alpha_5 d(x_n,Tx_{n-1})$
$\leq \alpha_1 d(x_{n-1},x_n)+\alpha_2 d(x_{n-1},x_n)+\alpha_3 d(x_n,x_{n+1})+\alpha_4 d(x_{n-1},x_{n+1})+\alpha_5 d(x_n,x_n)$

$$(1-\alpha_3)d_n \leq (\alpha_1+\alpha_2)d_{n-1}+\alpha_4 d^*_{n-1}. \tag{3.4}$$

If $d^*_{n-1} \leq d_{n-1}$, then from (3.4) we have
$(1-\alpha_3)d_n \leq (\alpha_1+\alpha_2+\alpha_4)d_{n-1}$
$d_n \leq (\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3})d_{n-1}$
$\leq (\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3})^2 d_{n-2}$
$\vdots$
$\leq (\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3})^n d_0$
which implies, $\lim_{n\to\infty} d_n = 0$.

If $d_{n-1} \leq d^*_{n-1}$, then from (3.4) we have
$(1-\alpha_3)d_n \leq (\alpha_1+\alpha_2+\alpha_4)d^*_{n-1}$

$$d_n \leq (\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3})d^*_{n-1}. \tag{3.5}$$

Again,
$d^*_{n-1} = d(x_{n-1},x_{n+1}) = d(Tx_{n-2},Tx_n)$
$\leq \alpha_1 d(x_{n-2},x_n)+\alpha_2 d(x_{n-2},Tx_{n-2})+\alpha_3 d(x_n,Tx_n)+\alpha_4 d(x_{n-2},Tx_n)+\alpha_5 d(x_n,Tx_{n-2})$
$\leq \alpha_1 d(x_{n-2},x_n)+\alpha_2 d(x_{n-2},x_{n-1})+\alpha_3 d(x_n,x_{n+1})+\alpha_4 d(x_{n-2},x_{n+1})+\alpha_5 d(x_n,x_{n-1})$.
Therefore,

$$\lim_{n\to\infty} d^*_{n-1} \leq \lim_{n\to\infty} \alpha_1 d^*_{n-2} + \lim_{n\to\infty} \alpha_4 d(x_{n-2},x_{n+1}). \tag{3.6}$$

If $d^*_{n-2} \leq d(x_{n-2},x_{n+1})$, then from (3.6), we get
$\lim_{n\to\infty} d^*_{n-1} \leq \lim_{n\to\infty}(\alpha_1+\alpha_4)d(x_{n-2},x_{n+1})$
$\leq \lim_{n\to\infty}(\alpha_1+\alpha_4)^2 d(x_{n-3},x_n)$
$\vdots$
$\leq \lim_{n\to\infty}(\alpha_1+\alpha_4)^{n-1} d(x_0,x_3)$
$= 0.$
Again, if $d(x_{n-2},x_{n+1}) \leq d^*_{n-2}$, then from (3.6), we get
$\lim_{n\to\infty} d^*_{n-1} \leq \lim_{n\to\infty}(\alpha_1+\alpha_4)d^*_{n-2}$
$\leq \lim_{n\to\infty}(\alpha_1+\alpha_4)^2 d^*_{n-3}$
$\vdots$
$\leq \lim_{n\to\infty}(\alpha_1+\alpha_4)^{n-1} d(x_0,x_2)$
$= 0.$
From (3.5), we get
$\lim_{n\to\infty} d_n \leq \lim_{n\to\infty}(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3})d^*_{n-1} = 0.$
Thus we have
$\lim_{n\to\infty} d_n = 0$ and $\lim_{n\to\infty} d^*_n = 0$.

Now we show $\{x_n\}$ is a Cauchy sequence.

$$\text{i.e., } \lim_{n\to\infty} d(x_{n+2v},x_{n+2v+j}) = 0. \tag{3.7}$$

To show this we apply Mathematical Induction on $j \in \mathbb{N}$.
Clearly (3.7) holds for $j = 1, 2$.

Suppose (3.7) holds for j.
So
$\lim_{n\to\infty} d(x_{n+2v}, x_{n+2v+j}) = 0$, $\lim_{n\to\infty} d(x_{n+3v}, x_{n+3v+j}) = 0$ and $\lim_{n\to\infty} d(x_{n+3v+1}, x_{n+3v+j+1}) = 0$.
We have to show
$\lim_{n\to\infty} d(x_{n+2v}, x_{n+2v+j+1}) = 0$.

Since
$d(x_{n+2v}, x_{n+2v+j+1})$
$\leq s[d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + \dots$
$+d(x_{n+2v+v}, x_{n+2v+v+1}) + d(x_{n+2v+v+1}, x_{n+2v+j+1})]$
$= s[d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + \dots + d(x_{n+2v+v}, x_{n+2v+v+1})]$
$+ sd(x_{n+2v+j+1}, x_{n+3v+1})$
$\leq s[d(x_{n+2v}, x_{n+2v+2}) + d(x_{n+2v+2}, x_{n+2v+3}) + \dots + d(x_{n+2v+v}, x_{n+2v+v+1})]$
$+s^2[d(x_{n+2v+j+1}, x_{n+2v+j+2})+d(x_{n+2v+j+2}, x_{n+2v+j+3})+\dots+d(x_{n+2v+j+v}, x_{n+2v+j+v+1})+$
$d(x_{n+2v+j+v+1}, x_{n+3v+1})]$
$= s[d^*_{n+2v} + d_{n+2v+2} + \dots + d_{n+3v}]$
$+ s^2[d_{n+2v+j+1} + d_{n+2v+j+2} + \dots + d_{n+3v+j} + d(x_{n+3v+j+1}, x_{n+3v+1})]$.
Therefore,
$\lim_{n\to\infty} d(x_{n+2v}, x_{n+2v+j+1}) = 0$. [since each of $d_n, d^*_n \to 0$ as $n \to \infty$].
Thus $\{x_n\}$ is a Cauchy sequence. Since the space is complete, there exists a $x \in X$ such that
$\lim_{n\to\infty} d(x_n, x) = 0$.
Now we show that x is a fixed point of T.
Since
$\lim_{n\to\infty} d(Tx, x) \leq \lim_{n\to\infty} s[d(Tx, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+v-1}, x)]$
$\leq \lim_{n\to\infty} sd(Tx, Tx_{n-1})$
$\leq \lim_{n\to\infty} s[\alpha_1 d(x, x_{n-1})+\alpha_2 d(x, Tx)+\alpha_3 d(x_{n-1}, Tx_{n-1})+\alpha_4 d(x, Tx_{n-1})+\alpha_5 d(x_{n-1}, Tx)]$
$= \lim_{n\to\infty} s[\alpha_1 d(x,x_{n-1})+\alpha_2 d(x, Tx)+\alpha_3 d(x_{n-1}, x_n)+\alpha_4 d(x, x_n)+\alpha_5 d(x_{n-1}, Tx)]$

$$\text{implies,}(1 - s\alpha_2) \lim_{n\to\infty} d(Tx, x) \leq \lim_{n\to\infty} \alpha_5 d(x_{n-1}, Tx) \tag{3.8}$$

Again,
$\lim_{n\to\infty} d(x_{n-1}, Tx)$
$\leq \lim_{n\to\infty} \alpha_1 d(x_{n-2}, x)+\alpha_2 d(x_{n-2}, Tx_{n-2})+\alpha_3 d(x, Tx)+\alpha_4 d(x_{n-2}, Tx)+\alpha_5 (x, Tx_{n-2})$
$= \lim_{n\to\infty} \alpha_1 d(x_{n-2}, x)+\alpha_2 d(x_{n-2}, x_{n-1})+\alpha_3 d(x, Tx)+\alpha_4 d(x_{n-2}, Tx)+\alpha_5 (x, x_{n-1})$

$$= \alpha_3 d(x, Tx) + \lim_{n\to\infty} \alpha_4 d(x_{n-2}, Tx) \tag{3.9}$$

If $d(Tx, x) \leq d(x_{n-2}, Tx)$, then from (3.9) we have
$\lim_{n\to\infty} d(x_{n-1}, Tx) \leq \lim_{n\to\infty}(\alpha_3 + \alpha_4) d(x_{n-2}, Tx)$
$\leq \lim_{n\to\infty}(\alpha_3 + \alpha_4)^2 d(x_{n-3}, Tx)$
$\vdots$
$\leq \lim_{n\to\infty}(\alpha_3 + \alpha_4)^{n-1} d(x_0, Tx)$
$= 0$.
Then from (3.8) we get,

$(1-s\alpha_2)\lim_{n\to\infty} d(Tx,x) \le \lim_{n\to\infty} \alpha_5 d(x_{n-1},Tx) = 0$
implies, $d(Tx,x)=0$ i.e., $Tx=x$.
Again if $d(x_{n-2},Tx) \le d(Tx,x)$, then from (3.9) we get, $\lim_{n\to\infty} d(x_{n-1},Tx) \le (\alpha_3+\alpha_4)d(x,Tx)$.
Then from (3.8) we get,
$(1-s\alpha_2)\lim_{n\to\infty} d(Tx,x) \le \lim_{n\to\infty} \alpha_5 d(x,Tx)$
implies, $d(Tx,x)=0$ i.e., $Tx=x$.
Thus x is a fixed point of T.

Now we show that x is unique.
Let, y be another fixed point of T.
Then
$d(x,y)=d(Tx,Ty)$
$\le \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(y,Ty)+\alpha_4 d(x,Ty)+\alpha_5 d(y,Tx)$
$= \alpha_1 d(x,y)+\alpha_2 d(x,x)+\alpha_3 d(y,y)+\alpha_4 d(x,y)+\alpha_5 d(y,x)$
i.e., $(1-\alpha_4-\alpha_5)d(x,y)\le 0$
implies, $d(x,y)=0$ i.e., $x=y$.
Thus the fixed point is unique.

Hence the theorem. □

Example 3.3. Let $X=C[0,1]$ and X' be given by $\{x\in C[0,1] : \sup_{0\le t\le 1}|x(t)| \le 1\}$. Let us define a function $d : X\to X$ by

$$d(x,y)=\begin{cases} \sup\limits_{0\le t\le 1}|x(t)-y(t)|, \ if\ x,y\in X'; \\ \dfrac{4}{5};\ \text{if } x,y\in X\setminus X'; \\ 2,\ \text{if any one of } x \text{ and } y \text{ lie in } X \text{and other is } X\setminus X'; \\ 0, otherwise. \end{cases}$$

Clearly, (X,d) is a $b_v(s)$ metric space with coefficient $s\ge 1$.

Now we consider a mapping $T : X\to X$ given by

$$(Tx)(t)=\begin{cases} \dfrac{t^2}{2}\,x(t),\ \text{if } x\in X'; \\ 0,\ \text{if } x\in X\setminus X'. \end{cases}$$

Any one can easily check that

$$d(Tx,Ty)<d(x,y)$$

for all $x,y\in X; x\ne y$. Now assume $x\in X$ and $\epsilon>0$ be arbitrary. Then for $\epsilon=\delta$ and $x\in X\setminus X'$

$$d(T^ix,T^jx)<\epsilon+\delta \Rightarrow d(T^{i+1}x,T^{j+1}x)\le\epsilon$$

for all $i,j\in\mathbb{N}\cup\{0\}$. Again suppose $x\in X'$ and $d(T^ix,T^jx)<\epsilon+\delta$, for some $i,j\in\mathbb{N}\cup\{0\}$.
Therefore,

$$\sup_{0\le t\le 1}|(\frac{t^2}{2})^i x(t)-(\frac{t^2}{2})^j y(t)|<\epsilon+\epsilon$$

$$\Rightarrow \sup_{0\le t\le 1} |(\frac{t^2}{2})^{i+1}x(t) - (\frac{t^2}{2})^{j+1}y(t)| < \epsilon$$
$$\Rightarrow d(T^{i+1}x, T^{j+1}x) < \epsilon.$$

Therefore, we see that all the conditions of the **Theorem 3.1** are satisfied. So T has a unique fixed point in X. Clearly, $x(t) = 0 \in C[0,1]$ for all $t \in [0,1]$ is the required fixed point of T.

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