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# UNIFORM DECAY RATES FOR NONLINEARLY DAMPED DISCRETIZATIONS

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Abstract. We study abstract dissipative systems in which the generator of a contraction semigroup, possibly including a numerical viscosity, is perturbed by a nonlinear feedback bounded near the origin by two increasing profiles. We prove that the decay rate dictated by the feedback follows from a linear hypothesis, the exponential decay of the associated linear damped system. No convexity, regularity or Lipschitz condition is required, and the constants do not depend on the norm of the observation operator. The result extends to the implicit midpoint scheme and yields mesh-independent decay rates for discretized nonlinear boundary feedback, an open case.

*Keywords.* Nonlinear damping, energy decay rates, uniform discretization, numerical viscosity, boundary feedback.
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## 1. Introduction

Let $X$ be a Hilbert space and consider the dissipative system

$$u'(t) + Au(t) + C^*G(u(t)) = 0, \qquad u(0) = u_0, \tag{1.1}$$

where $-A$ generates a contraction semigroup, $C$ is a linear observation operator with values in a space $U$ of functions on a measure space $(\Omega, \mu)$, and $G$ is a nonlinear feedback which behaves, pointwise, like $g(|Cu|)$ near the origin, $g$ being an increasing function that may vanish to arbitrary order at 0. This framework contains the wave, plate, Schrödinger and transport equations with localized nonlinear damping, their finite-difference semi-discretizations, and boundary feedbacks after discretization.

Since the work of Lasiecka–Tataru [17] and Martinez [19], many methods have been devised to obtain decay rates for arbitrarily growing feedbacks. The optimal-weight convexity method of Alabau-Boussouira [3, 4] produces a one-step formula for the decay rate in terms of $g$, which is optimal in finite dimension [4], and extends to indirectly damped coupled systems [8]. In [5, 6], Alabau-Boussouira

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and Ammari combined this method with an indirect argument: the nonlinear rate follows from the observability inequality of the associated conservative linear system. Alabau-Boussouira, Privat and Trélat [7] extended this to first-order systems with bounded damping $u'+\mathcal{A}u+BF(u)=0$ ($\mathcal{A}$ skew-adjoint, $B$ bounded) and to space, time and full discretizations with numerical viscosity, building on the linear theory of Ervedoza and Zuazua [14] and on the uniform stabilization results of Tcheugoué Tébou and Zuazua [25, 26] and Münch and Pazoto [21]. Controllability and stabilization of abstract evolution systems remain an active topic; see, e.g., [9] for semilinear systems with memory.

1.1. **Two limitations of the existing approach.** The argument of [7] consists of two comparisons: nonlinear system $\to$ linear damped system (Lemma 1 there), then linear damped system $\to$ conservative system (Lemma 2 there), followed by the observability inequality and a weighted Jensen argument. Two features limit its scope.

The first limitation is that the constants depend on $\|C\|$. The second comparison produces constants that grow like $T^2\|B\|^2$. Hence the method cannot reach boundary feedback, even after space discretization: there the discrete observation operators $C_h$ satisfy $\|C_h\| \to \infty$ as $h \to 0$. This case is listed among the open problems of [7, Section 4]. In the time and full discretizations the second comparison is also the delicate step, and the full discretization [7, Theorem 4] is only sketched.

The second limitation concerns convexity and regularity. Like all optimal-weight results, [7] requires $g \in C^1$ with $g'(0)=0$, strict convexity of $H(x)=\sqrt{x}\,g(\sqrt{x})$, the condition $sg'(s)^2/g(s)\to 0$, and a locally Lipschitz feedback. Many natural feedbacks violate these conditions: monotone feedbacks that are not Lipschitz at 0 (e.g. $|v|^{\gamma-1}v$, $\gamma<1$, near 0), oscillating profiles such as $g(s)=\frac{1}{4}s^2(3+\sin(5\ln\frac{1}{s}))$ (Figure 2), and two-sided bounds with different profiles above and below.

1.2. **The two observations of this paper.** The first observation is that the first comparison suffices. If the hypothesis is formulated on the linear *damped* system rather than on the conservative one, the second comparison disappears. If $z'+(A+C^*C)z=0$, $z(0)=u_0$, satisfies $\|z(T)\|^2 \le (1-\delta)\|z(0)\|^2$, then (Lemma 4.5)

$$\frac{\delta}{2}\|u_0\|^2 \le 2\int_0^T q(u)\,\mathrm{d}t + 4\int_0^T \big(\|Cu\|_U^2+\|G(u)\|_U^2\big)\,\mathrm{d}t, \qquad q(x)=\mathrm{Re}\langle Ax,x\rangle,$$

with no constant depending on $\|C\|$, on $T$ or on $X$. This is precisely the kind of statement proved in the numerical-analysis literature, where uniform exponential stability of viscous linear schemes is established [25, 26, 21, 14]. The comparison has an exact discrete counterpart for the implicit midpoint rule with viscosity (Lemma 5.4), which works for unbounded viscosity operators.

The second observation is that a level-set inequality can replace convexity. Let

$$h(\eta) := \min_{i=1,2}\ \min_{s\in[\eta,1]} \frac{g_i(s)}{s} \qquad (0<\eta\le 1),$$

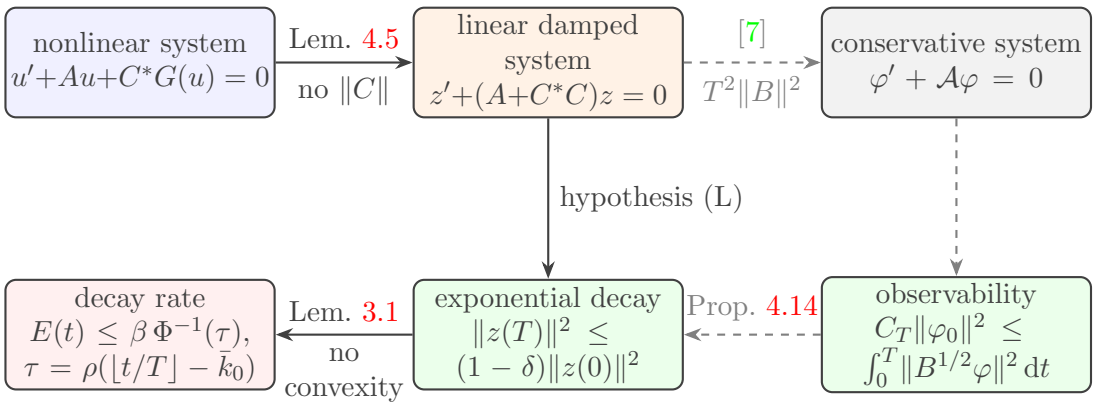



FIGURE 1. Logical structure. The approach of [7] (dashed) passes through the conservative system and the observability inequality, which costs a factor $T^2\|B\|^2$. We stop at the linear damped system: its exponential decay is the only linear input, and the comparison with it involves no constant depending on $\|C\|$. Observability still implies our hypothesis (Proposition 4.14), so nothing is lost.

the nondecreasing lower envelope of $g_i(s)/s$ (Figure 2). At every point of the damping region and for every level $\eta\in(0,1]$,

$$h(\eta)\big(|f|^2+|r|^2\big)\le c_A\ \mathrm{Re}(r\cdot\bar{f})+c_B\,\eta^2 h(\eta),$$

where $f$ is the observed quantity and $r$ the feedback (Lemma 3.1). This replaces both the weighted Jensen inequality and the convex conjugate. Choosing $\eta^2$ proportional to the current energy makes the error term absorbable, and one obtains the one-step inequality

$$E(s+T)\le E(s)-\rho\,E(s)\,h\Big(\min\Big\{1,\sqrt{E(s)/\beta}\Big\}\Big). \tag{1.2}$$

1.3. **Main results.** The abstract continuous-time result is Theorem 4.6. Under the linear hypothesis above, a pointwise structure hypothesis on $G$ involving two arbitrary profiles $g_1,g_2$, and a mild well-posedness hypothesis, every solution satisfies

$$E(t)\le\beta\,\Phi^{-1}\big(\rho(\lfloor t/T\rfloor-\bar{k}_0)\big),\qquad \Phi(x)=\int_x^1\frac{\mathrm{d}y}{y\,h(\sqrt{y})},\qquad \bar{k}_0\lesssim 1+\ln^+\frac{E(0)}{\beta}.$$

The constants $\beta,\rho$ depend only on $(\delta,T)$, on $m=\int_\Omega b\,\mathrm{d}\mu$ (the size of the damping region) and on the structural constants; the data enter only through the logarithmic shift $\bar{k}_0$. When the functions $g_i(s)/s$ are nondecreasing, $\Phi(x)=2\Theta(\sqrt{x})$ with $\Theta(z)=\int_z^1\mathrm{d}\sigma/\min_i g_i(\sigma)$, so that the energy decays like $z(t)^2$, where $z'=-c\,g(z)$. Corollaries 4.8–4.11 give uniform exponential decay when $\inf g_i(s)/s>0$, the rate $E(t)\lesssim t^{-2/(p-1)}$ when $g_i(s)\ge cs^p$ without any monotonicity or convexity, the stretched exponential $e^{-c\sqrt{t}}$ for $g(s)=s/(1+\ln\frac{1}{s})$, and $(\ln t)^{-2}$ for $g(s)=e^{-1/s}$ (Figure 4). The bound is never worse than convexification: our dissipation function dominates the convex minorant of $x\mapsto x\min_i\ell_i(\sqrt{x})$ (Proposition 4.13), and

under the hypotheses of [7] with $\Lambda_H \le \lambda < 1$ it implies the rate $(H')^{-1}(c/t)$ of [7, Theorem 1], with explicit constants (Proposition 4.12 and Figure 5).

The discrete-time counterpart is Theorem 5.5: the same conclusion holds for the implicit midpoint scheme with any nonnegative self-adjoint viscosity, uniformly with respect to all parameters on which $(\delta, T, m)$ do not depend (Example 5.6 and Figure 6). The observability hypotheses of [7] imply ours with explicit $\delta$ (Propositions 4.14 and 5.7), so that Theorems 1–3 of [7] are contained in Theorems 4.6 and 5.5; combined with [14, Theorem 1.1], this yields a complete proof of uniform decay for fully discrete schemes with nonlinear localized damping (Corollary 6.6). For nonlinear boundary feedback, space semi-discretizations decay at the rate dictated by $g$, uniformly in the mesh size, as soon as the corresponding linear viscous scheme is uniformly exponentially stable, as in [26] (Corollary 6.8, illustrated numerically in Figure 11).

We also point out that hypotheses (34)–(35) of [7] fail for $P_1$ finite elements (Remark 6.5 and Figure 9); we replace them by a nodal structure hypothesis, which holds for finite differences and lumped-mass finite elements. Section 6.2 treats heterogeneous localized damping with variable exponents, mixing sublinear (non-Lipschitz) and superlinear behaviour (Example 6.1), and an oscillator with an oscillating feedback, for which the rate $t^{-2}$ is attained although no convexity holds (Example 6.2 and Figure 8); Figure 14 summarizes which situations are covered by [7] and by the present paper. Finally, in Section 7 every constant is computed for the damped string and its Galerkin approximations, starting from the closed-form linear estimate of Proposition 7.1. The exponents are attained, including for non-Lipschitz feedbacks, where the upper profile $g_2$ governs the rate, but the constants are not; feedbacks with thin dips show that pointwise methods can lose the exponent (Example 7.4).

1.4. **Relation to previous work.** Taking the exponential stability of the linear closed loop as the basic hypothesis for nonlinearly damped systems $z' = Az - B\sigma(B^*z)$ is already done by Marx, Chitour and Prieur [20], who construct Lyapunov functionals; their nonlinearities, however, behave linearly near the origin (saturations), and no decay rates for feedbacks vanishing at higher order are derived there. Paunonen and Seifert [23] study the same structure with nonuniform observability and obtain polynomial rates. The decay rates themselves are, up to constants, those of Lasiecka–Tataru [17] and Alabau-Boussouira [3, 4]; their optimality in finite dimension is due to [4]. The comparison of the nonlinear system with the linear damped one is Lemma 1 of [7]. The equivalence between observability and exponential stability for bounded damping is classical [16]. Our contribution is to show that this comparison alone carries the whole proof, with constants independent of $\|C\|$ and of the data, that convexity can be replaced by a monotone envelope at no cost, and to draw the consequences above.

The companion paper [1] develops a different convexity-free argument, a *halving principle* with an energy-dependent time step, and applies it to continuous boundary damping in star-shaped domains, where $C$ is unbounded and the comparison of Lemma 4.5 is not available in our abstract form (Section 8). It also proves

lower bounds that match the upper bounds in a weakly nonlinear regime. The two papers overlap only in the explicit rates for localized damping (Section 4.1), which we keep here because they are needed for the discrete statements. For the one-dimensional wave equation in $W_0^{1,p} \times L^p$, see [2].

**Notation.** $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Inner products are linear in the first argument. $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$ (floor function), $\lceil x \rceil$ the smallest integer not less than $x$ and $\ln^+ := \max\{\ln, 0\}$. For a measure space $(\Omega, \mu)$ and $d \geq 1$ we write $L^2(\Omega, \mu; \mathbb{K}^d)$ with the Euclidean norm $|\cdot|$ on $\mathbb{K}^d$.

## 2. Setting and hypotheses

### 2.1. The linear part.

**Hypothesis 2.1** (A). $X$ is a Hilbert space over $\mathbb{K}$ and $A : D(A) \subset X \to X$ is a linear operator such that $-A$ generates a $C_0$-semigroup of contractions on $X$. We set

$$q(x) := \mathrm{Re}\langle Ax, x\rangle \geq 0, \qquad x \in D(A).$$

Typically $A = \mathcal{A} + Q$ with $\mathcal{A}$ skew-adjoint and $Q \geq 0$ a (numerical) viscosity, in which case $q(x) = \langle Qx, x\rangle$. Since $(x, y) \mapsto \frac{1}{2}(\langle Ax, y\rangle + \langle x, Ay\rangle)$ is a nonnegative Hermitian form on $D(A)$ whose quadratic form is $q$, the Cauchy–Schwarz inequality gives

$$q(x + y) \leq 2q(x) + 2q(y), \qquad x, y \in D(A). \tag{2.1}$$

### 2.2. The feedback.

**Hypothesis 2.2** (g). $g_1, g_2 : [0, 1] \to [0, \infty)$ are continuous and nondecreasing, with $g_i(0) = 0$, $g_i(s) > 0$ and $g_i(s) \leq s$ for $s \in (0, 1]$.

The normalization $g_i(s) \leq s$ costs nothing: replacing $g_i$ by $\min\{g_i(s), s\}$ only weakens the conditions (P2) below. No regularity, no strict monotonicity, no convexity and no monotonicity of $g_i(s)/s$ is assumed.

**Hypothesis 2.3** (P). (P0) $(\Omega, \mu)$ is a measure space, $U = L^2(\Omega, \mu; \mathbb{K}^d)$, $C \in \mathcal{L}(X, U)$, and $G : X \to U$ is continuous. There is a measurable $b : \Omega \to [0, \infty)$ with $0 < m := \int_\Omega b \, \mathrm{d}\mu < \infty$. For every $x \in X$ there are measurable $f = f_x$, $r = r_x : \Omega \to \mathbb{K}^d$ such that $Cx = \sqrt{b}\, f$ and $G(x) = \sqrt{b}\, r$ $\mu$-a.e.

There are constants $c_0 \in (0, 1]$, $c_1, c_2 > 0$ and $\varepsilon_0 \in (0, 1]$ such that, for $\mu$-a.e. $\omega$ with $b(\omega) > 0$:

(P1) $\mathrm{Re}(r \cdot \bar{f}) \geq c_0 |f||r|$;
(P2) if $|f| \leq \varepsilon_0$, then $c_1 g_1(|f|) \leq |r| \leq c_2$ and $g_2(|r|/c_2) \leq |f|$;
(P3) if $|f| > \varepsilon_0$, then $c_1|f| \leq |r| \leq c_2|f|$.

*Remark* 2.4. Hypothesis (P) contains the hypotheses of [7] and of the optimal-weight method. There one assumes $c_1 g(|f|) \leq |r| \leq c_2 g^{-1}(|f|)$ for $|f| \leq \varepsilon_0$, with $g$ increasing. This implies (P2) with $g_1 = g_2 = \min\{g, \mathrm{id}\}$, after shrinking $\varepsilon_0$ so that $g^{-1}(\varepsilon_0) \leq 1$: indeed $|r|/c_2 \leq g^{-1}(|f|)$ gives both $|r| \leq c_2$ and $g(|r|/c_2) \leq |f|$.

Shrinking $\varepsilon_0$ only changes $c_1, c_2$, because on $\{\varepsilon_0' < |f| \le \varepsilon_0\}$ the bounds in (P2) imply linear bounds.

Bounded damping as in [7] also fits this framework. Let $B \ge 0$ be bounded and self-adjoint on a separable $X$. By the spectral theorem there is a unitary $\mathsf{U} : L^2(\Omega,\mu) \to X$, with $\mu$ a probability measure, such that $\mathsf{U}^{-1}B\mathsf{U}$ is multiplication by $b \ge 0$. Taking $U = L^2(\Omega,\mu)$, $C = B^{1/2}$ (read through $\mathsf{U}$), $f = \mathsf{U}^{-1}x$, $r = \varrho(f) = \mathsf{U}^{-1}F(\mathsf{U}f)$ and $G(x) = \sqrt{b}\,\varrho(f)$, we get $C^*G(x) = BF(x)$ and $m \le \|B\|$; (P1)–(P3) with $c_0 = 1$ are [7, (5)–(6)].

Vector-valued and complex feedbacks are allowed: for the Schrödinger feedback $r = f\varrho(x,|f|)$ with $\varrho \ge 0$, (P1) holds with $c_0 = 1$. For boundary feedback, discretized or not, take $\Omega$ to be the controlled part of the boundary, or a single boundary node, $b \equiv 1$, $f = Cx$ the boundary velocity and $r = \varrho(f)$; then $m = \mu(\Omega)$.

Finally, using two different profiles $g_1 \ne g_2$ costs nothing and decouples the lower bound on the feedback, which controls $|f|^2$, from the upper bound, which controls $|r|^2$. For instance $|r| \sim |f|^\gamma$ with $\gamma \in (0,1)$ near 0 (a monotone feedback that is not Lipschitz) satisfies (P2) with $g_1(s) = s$ and $g_2(s) = s^{1/\gamma}$.

### 2.3. **The decay modulus.**

**Definition 2.5.** For $i = 1,2$ and $s, \eta \in (0,1]$ let $\ell_i(s) := g_i(s)/s$ and

$$h(\eta) := \min_{i=1,2} \min_{s\in[\eta,1]} \ell_i(s), \qquad h(0) := 0,$$

$$\kappa_0 := h(1) = \min\{g_1(1), g_2(1)\}, \qquad g_\flat := \min\{g_1, g_2\}.$$

**Lemma 2.6.** *The function $h$ is continuous, positive and nondecreasing on $(0,1]$, and $g_\flat(\eta) \le h(\eta) \le \kappa_0 \le 1$ for $\eta \in (0,1]$. If $\ell_1, \ell_2$ are nondecreasing, then $h(\eta) = g_\flat(\eta)/\eta$.*

*Proof.* Each $\ell_i$ is continuous and positive on $(0,1]$; hence $\eta \mapsto \min_{[\eta,1]} \ell_i$ is positive, nondecreasing and continuous. For $s \in [\eta,1]$, $\ell_i(s) \ge g_i(s) \ge g_i(\eta)$, so $h(\eta) \ge g_\flat(\eta)$; and $h \le h(1) = \kappa_0 \le 1$ by monotonicity and $g_i(1) \le 1$. If the $\ell_i$ are nondecreasing, $\min_{[\eta,1]} \ell_i = \ell_i(\eta)$. □

Let $\beta > 0$. We use the functions

$$\begin{aligned} \Psi_\beta(e) &:= e\,h\Big(\min\Big\{1, \sqrt{e/\beta}\Big\}\Big) \quad (e \ge 0), \\ M(y) &:= y\,h(\sqrt{y}), \qquad \Phi(x) := \int_x^1 \frac{\mathrm{d}y}{M(y)} \quad (0 < x \le 1). \end{aligned} \tag{2.2}$$

By Lemma 2.6, $M$ is continuous, nondecreasing and positive on $(0,1]$, with $M(y) \le y$; hence $\Phi$ is a continuous decreasing bijection of $(0,1]$ onto $[0,\infty)$,

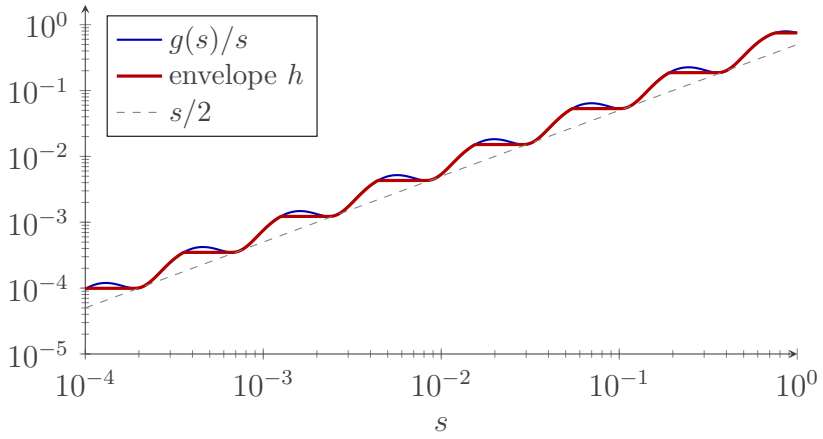



FIGURE 2. The oscillating profile $g(s) = \frac{1}{4}s^2\big(3 + \sin(5\ln\frac{1}{s})\big)$ of Example 4.10: $g(s)/s$ is not monotone on any interval $(0,\eta)$, and $\sqrt{x}\,g(\sqrt{x})$ is not convex on any interval $(0,x_0)$, so the optimal-weight convexity method does not apply to $g$ directly. Our rate uses only the monotone envelope $h(\eta) = \min_{[\eta,1]} g(s)/s$ (thick), which is built into the statements; here $h(\eta) \geq \eta/2$ and the decay is $t^{-2}$.

and we write $\Phi^{-1} : [0,\infty) \to (0,1]$. The substitution $y = \sigma^2$ gives

$$\Phi(x) = 2\int_{\sqrt{x}}^{1} \frac{\mathrm{d}\sigma}{\sigma\, h(\sigma)}; \qquad \text{if the } \ell_i \text{ are nondecreasing,} \quad \Phi(x) = 2\,\Theta(\sqrt{x}), \quad \Theta(z) := \int_z^1 \frac{\mathrm{d}\sigma}{g_\flat(\sigma)}. \tag{2.3}$$

Since $z(t) = \Theta^{-1}(ct)$ solves the amplitude equation $z' = -c\,g_\flat(z)$, the rate $\Phi^{-1}(ck) = \Theta^{-1}(ck/2)^2$ is the square of the amplitude of the model ODE.

## 3. Two elementary lemmas

The first lemma replaces the weighted Jensen argument of [7, Lemma 3], [4] and the Fenchel–Young inequality for the convex function $H$: a choice of level suffices, pointwise. It is illustrated in Figure 3.

**Lemma 3.1.** *Assume* (g) *and* (P), *and set*

$$c_A := \frac{c_1^{-1} + c_2}{c_0}, \qquad c_B := 1 + c_2^2.$$

*Then for every* $x \in X$ *and every* $\eta \in (0,1]$,

$$h(\eta)\big(\|Cx\|_U^2 + \|G(x)\|_U^2\big) \leq c_A\,\mathrm{Re}\langle G(x), Cx\rangle_U + c_B\,m\,\eta^2 h(\eta). \tag{3.1}$$

*In particular* $\mathrm{Re}\langle G(x), Cx\rangle_U \geq 0$. *If moreover* (P3) *holds for all values of* $|f|$, *then* $\|Cx\|^2 + \|G(x)\|^2 \leq c_A\,\mathrm{Re}\langle G(x), Cx\rangle$.

*Proof.* We have $\|Cx\|^2 + \|G(x)\|^2 = \int_\Omega b(|f|^2 + |r|^2)\,\mathrm{d}\mu$ and $\mathrm{Re}\langle G(x), Cx\rangle = \int_\Omega b\,\mathrm{Re}(r\cdot\bar f)\,\mathrm{d}\mu$. It therefore suffices to show that at $\mu$-a.e. point where $b>0$

$$h(\eta)\big(|f|^2+|r|^2\big) \le (c_1^{-1}+c_2)|f||r| + c_B\,\eta^2 h(\eta), \tag{3.2}$$

and then to use (P1) and integrate against $b\,\mathrm{d}\mu$. Recall $h\le 1$.

Consider first the term $|f|^2$. If $|f|\le\eta$, then $h(\eta)|f|^2\le\eta^2 h(\eta)$. If $\eta<|f|\le\varepsilon_0$, then $|f|\in[\eta,1]$ and, by (P2), $|f||r|\ge c_1|f|g_1(|f|)=c_1|f|^2\ell_1(|f|)\ge c_1h(\eta)|f|^2$. If $|f|>\varepsilon_0$, then by (P3) $h(\eta)|f|^2\le|f|^2\le c_1^{-1}|f||r|$.

Consider next the term $|r|^2$. If $|f|>\varepsilon_0$, then by (P3) $h(\eta)|r|^2\le|r|^2\le c_2|f||r|$. If $|f|\le\varepsilon_0$, put $\sigma:=|r|/c_2\in[0,1]$, so that $g_2(\sigma)\le|f|$ by (P2). If $\sigma\le\eta$, then $h(\eta)|r|^2\le c_2^2\eta^2h(\eta)$. If $\sigma>\eta$, then $\sigma\in(\eta,1]$ and

$$|f||r| = c_2\sigma|f| \ge c_2\sigma g_2(\sigma) = c_2\sigma^2\ell_2(\sigma) \ge c_2\sigma^2 h(\eta) = c_2^{-1}h(\eta)|r|^2.$$

Adding the worst case of each list gives (3.2). The last assertion is the case where only the alternatives $|f|>\varepsilon_0$ occur, with $h$ replaced by 1. □

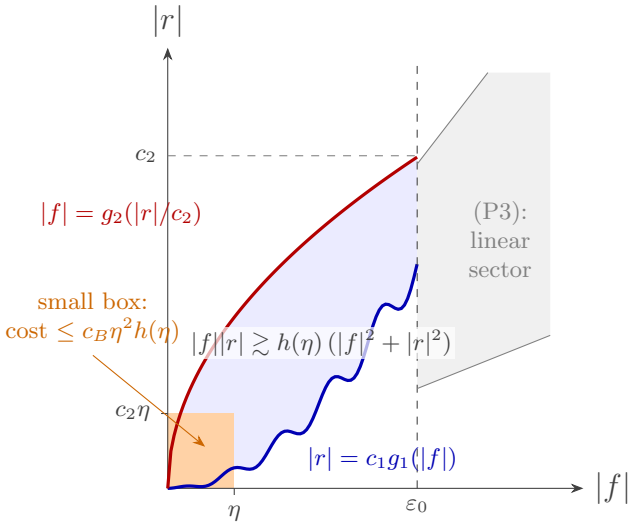



FIGURE 3. Lemma 3.1. The pair $(|f|,|r|)$ lies between the curves $|r|=c_1g_1(|f|)$ and $|f|=g_2(|r|/c_2)$ for $|f|\le\varepsilon_0$, and in a linear sector beyond. Outside the small box of size $\eta$ (shaded), the dissipation $|f||r|$ controls $h(\eta)(|f|^2+|r|^2)$, because the slopes $g_i(s)/s$ on $[\eta,1]$ are at least $h(\eta)$. Inside the box one pays $\eta^2h(\eta)$. Choosing $\eta^2$ proportional to the energy makes this cost absorbable, without any convexity.

**Lemma 3.2.** *Let $\varrho\in(0,1]$ and let $(e_k)_{k\ge k_0}\subset[0,1]$ satisfy $e_{k+1}\le e_k-\varrho M(e_k)$ for all $k\ge k_0$. Then $e_k\le\Phi^{-1}(\varrho(k-k_0))$ for all $k\ge k_0$.*

*Proof.* The sequence is nonincreasing. If $e_k=0$ the claim is trivial and all later terms vanish. Otherwise, since $M$ is nondecreasing,

$$\Phi(e_{j+1})-\Phi(e_j)=\int_{e_{j+1}}^{e_j}\frac{\mathrm{d}y}{M(y)}\ge\frac{e_j-e_{j+1}}{M(e_j)}\ge\varrho.$$

Hence $\Phi(e_k) \geq \Phi(e_{k_0}) + \varrho(k-k_0) \geq \varrho(k-k_0)$, and $\Phi$ is decreasing. □

## 4. Continuous time

**Hypothesis 4.1** (L)**.** There exist $T > 0$ and $\delta \in (0,1]$ such that every solution of $z' + (A + C^*C)z = 0$ satisfies $\|z(T)\|^2 \leq (1-\delta)\|z(0)\|^2$.

Here $A + C^*C$ is a bounded accretive perturbation of $A$, so $-(A+C^*C)$ generates a contraction semigroup [24, Chapter 3]. Hypothesis (L) is equivalent to exponential stability of that semigroup. It concerns the linear closed loop only, for which exponential stability can be established by spectral, multiplier, Lyapunov or integral-transformation (backstepping) methods; see, e.g., [11, 16] and, for a recent constructive example with boundary control of a one-dimensional wave equation, [13].

**Hypothesis 4.2** (W)**.** There are a dense set $D \subset D(A)$ and maps $\mathcal{S}(t) : X \to X$, $t \geq 0$, each continuous, with $\mathcal{S}(0) = I$, such that for $u_0 \in D$ the function $u(t) = \mathcal{S}(t)u_0$ satisfies: $u \in W^{1,1}_{\mathrm{loc}}([0,\infty); X)$; $u(t) \in D(A)$ for every $t \geq 0$; and $u' + Au + C^*G(u) = 0$ a.e. on $(0,\infty)$. For $u_0 \in X$ we call $\mathcal{S}(\cdot)u_0$ the (generalized) solution.

**Proposition 4.3.** *Assume* (A) *and* (P0) *with* $G(0) = 0$*, and let* $\mathcal{N} := C^*G$*. Then* (W) *holds with* $D = D(A)$ *under each of the following conditions:* (a) $\mathcal{N}$ *is monotone on* $X$*, i.e.* $\mathrm{Re}\langle \mathcal{N}(x) - \mathcal{N}(y), x - y\rangle \geq 0$*;* (b) $\mathcal{N}$ *is Lipschitz on bounded subsets of* $X$*, and* $\mathrm{Re}\langle \mathcal{N}(x), x\rangle \geq 0$ *for all* $x$*;* (c) $\dim X < \infty$*,* $\mathrm{Re}\langle \mathcal{N}(x), x\rangle \geq 0$ *for all* $x$*, and solutions of* $u' + Au + \mathcal{N}(u) = 0$ *are unique (e.g.* $\mathcal{N}$ *locally Lipschitz or monotone). Under* (g) *and* (P)*,* $G(0) = 0$ *and* $\mathrm{Re}\langle \mathcal{N}(x), x\rangle = \mathrm{Re}\langle G(x), Cx\rangle \geq 0$ *hold automatically.*

*Proof.* (a) $A$ is maximal monotone, and a monotone continuous everywhere-defined operator is maximal monotone; the sum of a maximal monotone operator and a monotone hemicontinuous everywhere-defined operator is maximal monotone [12, Chapter II]. So $A + \mathcal{N}$ is maximal monotone with domain $D(A)$. By the Kōmura–Kato–Brezis theorem [12, Chapter III], for $u_0 \in D(A)$ there is a unique Lipschitz $u$ with $u(t) \in D(A)$ for all $t$, satisfying the equation a.e.; the resulting semigroup is nonexpansive, hence continuous.

(b) For $u_0 \in X$ there is a unique maximal mild solution [24, Thm. 6.1.4]; for $u_0 \in D(A)$ it is a strong solution and is Lipschitz on compact subintervals [24, Thm. 6.1.6], since $X$ is reflexive. Then $\mathsf{h} = \mathcal{N}(u)$ is Lipschitz in time with values in $X$, hence $\mathsf{h} \in W^{1,1}(0,\tau; X)$. Writing $S_0(\cdot)$ for the semigroup generated by $-A$,

$$\int_0^t S_0(t-s)\mathsf{h}(s)\,\mathrm{d}s = \Big(\int_0^t S_0(r)\,\mathrm{d}r\Big)\mathsf{h}(0) + \int_0^t \Big(\int_0^{t-\sigma} S_0(r)\,\mathrm{d}r\Big)\mathsf{h}'(\sigma)\,\mathrm{d}\sigma.$$

Since $\int_0^\tau S_0(r)\,\mathrm{d}r$ maps $X$ into $D(A)$ with $A\int_0^\tau S_0(r)\,\mathrm{d}r = I - S_0(\tau)$ uniformly bounded, the closedness of $A$ shows that the convolution lies in $D(A)$ for every $t$. As $S_0(t)u_0 \in D(A)$, $u(t) \in D(A)$ for all $t$. Along such solutions $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \leq 0$; by density and continuous dependence (Gronwall on bounded sets), $\|u(t)\| \leq \|u_0\|$ for every mild solution, so solutions are global and $u_0 \mapsto u(t)$ is continuous.

(c) Peano's theorem gives local $C^1$ solutions, which are global because $\|u(t)\|$ is nonincreasing; with uniqueness, continuous dependence follows from the standard compactness argument for ODEs.

For the last claim, take $x = 0$: $Cx = 0$, so $f = 0$ where $b > 0$, and (P2) gives $c_1 g_1(0) \le |r|$ and $g_2(|r|/c_2) \le 0$, hence $r = 0$ because $g_2 > 0$ on $(0, 1]$; i.e. $G(0) = 0$. The sign follows from Lemma 3.1. □

*Remark* 4.4. If $\dim X < \infty$ and $\mathcal{N}$ is merely continuous with $\mathrm{Re}\langle \mathcal{N}(x), x\rangle \ge 0$, solutions exist globally but need not be unique, so (W) may fail as stated. The proofs of (i) and (ii) in Theorem 4.6, before the final density argument, use only that the given trajectory is $C^1$ with values in $D(A) = X$; hence the conclusions of Theorem 4.6 hold for every $C^1$ solution in this case.

**Lemma 4.5.** *Assume* (A), (P0), (P1) *and* (W), *and let* $u_0 \in D$, $u = \mathcal{S}(\cdot)u_0$. *Let* $s \ge 0$ *and let* $z$ *solve* $z' + (A + C^*C)z = 0$ *on* $[s, s + T]$ *with* $z(s) = u(s)$. *Then*

$$\int_s^{s+T} \big(q(z) + \|Cz\|^2\big)\,\mathrm{d}t \le 2\int_s^{s+T} q(u)\,\mathrm{d}t + 4\int_s^{s+T} \big(\|Cu\|^2 + \|G(u)\|^2\big)\,\mathrm{d}t. \quad (4.1)$$

*Proof.* Take $s = 0$. Since $u(0) \in D(A)$, $z$ is a classical solution. Put $\psi = u - z$. Then $\psi \in W^{1,1}(0, T; X)$, $\psi(t) \in D(A)$, $\psi(0) = 0$ and $\psi' = -A\psi - C^*G(u) + C^*Cz$ a.e. Hence, a.e. in $t$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2 = -q(\psi) - \mathrm{Re}\langle G(u) - Cz, Cu - Cz\rangle_U.$$

By (P1), $\mathrm{Re}\langle G(u), Cu\rangle \ge 0$, so

$$\begin{aligned}\mathrm{Re}\langle G(u) - Cz, Cu - Cz\rangle &\ge \|Cz\|^2 - \|Cz\|\big(\|G(u)\| + \|Cu\|\big)\\ &\ge \tfrac{1}{2}\|Cz\|^2 - \|G(u)\|^2 - \|Cu\|^2.\end{aligned}$$

Integrating and using $\psi(0) = 0$,

$$\int_0^T \big(q(\psi) + \tfrac{1}{2}\|Cz\|^2\big)\,\mathrm{d}t \le \mathsf{S} := \int_0^T \big(\|G(u)\|^2 + \|Cu\|^2\big)\,\mathrm{d}t.$$

By (2.1), $q(z) \le 2q(u) + 2q(\psi)$. Therefore $\int_0^T q(z) \le 2\int_0^T q(u) + 2\mathsf{S}$ and $\int_0^T \|Cz\|^2 \le 2\mathsf{S}$, which gives (4.1). □

**Theorem 4.6.** *Assume* (A), (g), (P), (L) *and* (W). *Set*

$$K := \max\{2, 4c_A\}, \qquad \rho := \frac{\delta}{2K}, \qquad \beta := \frac{8c_B m T}{\delta},$$

*and for* $u_0 \in X$ *let* $E(t) := \frac{1}{2}\|\mathcal{S}(t)u_0\|^2$. *Then* $E$ *is nonincreasing, and the following assertions hold.*

(i) *For every* $s \ge 0$, *the one-step inequality*

$$E(s + T) \le E(s) - \rho\,\Psi_\beta\big(E(s)\big) \quad (4.2)$$

*holds.*

(ii) *Let* $\kappa_0 = h(1)$ *and*

$$\bar{k}_0 := \left\lceil \frac{\ln^+(E(0)/\beta)}{\ln \frac{1}{1-\rho\kappa_0}} \right\rceil,$$

*Then* $E(kT) \le (1-\rho\kappa_0)^k E(0)$ *for* $0 \le k \le k_0 := \min\{k \ge 0 : E(kT) \le \beta\}$, *where* $k_0 \le \bar{k}_0$, *and*

$$E(t) \le \beta\, \Phi^{-1}\big(\rho(\lfloor t/T \rfloor - \bar{k}_0)\big) \qquad \text{whenever } \lfloor t/T \rfloor \ge \bar{k}_0. \tag{4.3}$$

(iii) *If* $\kappa_* := \inf_{0<\eta\le 1} h(\eta) > 0$, *then* $E(t) \le (1-\rho\kappa_*)^{\lfloor t/T \rfloor} E(0)$ *for all* $t \ge 0$; *if* (P3) *holds for all values of* $|f|$, *then* $E(t) \le (1-\delta/K)^{\lfloor t/T \rfloor} E(0)$. *The constants* $K, \rho, \beta$ *depend only on* $\delta, T, m, c_0, c_1, c_2$, *and* $h$ *depends only on* $g_1, g_2$. *Nothing depends on* $X$, $U$, $A$, $\|C\|$ *or* $\|G\|$, *and the initial data enter only through* $\bar{k}_0$.

*Proof.* We first prove (i) for $u_0 \in D$. Let $u_0 \in D$. Since $u \in W^{1,1}_{\mathrm{loc}}$, a.e. in $t$

$$E'(t) = \mathrm{Re}\langle u', u\rangle = -q(u) - \mathrm{Re}\langle G(u), Cu\rangle \le 0,$$

by Lemma 3.1. Hence, for $s \ge 0$,

$$E(s) - E(s+T) = \int_s^{s+T} \big(q(u) + \mathrm{Re}\langle G(u), Cu\rangle\big)\, \mathrm{d}t. \tag{4.4}$$

Next we derive a weighted one-step inequality. Let $s \ge 0$ with $E(s) > 0$, and let $z$ be as in Lemma 4.5. By (L) applied on $[s, s+T]$ (the equation is autonomous),

$$\delta E(s) \le \tfrac{1}{2}\|z(s)\|^2 - \tfrac{1}{2}\|z(s+T)\|^2 = \int_s^{s+T} \big(q(z) + \|Cz\|^2\big)\, \mathrm{d}t.$$

Let $\eta \in (0,1]$ and $w := h(\eta) \in (0,1]$. Multiply by $w$, use (4.1), $w \le 1$ in front of $q(u)$, and Lemma 3.1 at each time:

$$\delta w E(s) \le 2\int_s^{s+T} q(u) + 4c_A \int_s^{s+T} \mathrm{Re}\langle G(u), Cu\rangle + 4c_B m T \eta^2 h(\eta).$$

By (4.4),

$$\delta\, h(\eta) E(s) \le K\big(E(s) - E(s+T)\big) + 4c_B m T\, \eta^2 h(\eta). \tag{4.5}$$

It remains to choose the level. Let $\eta := \min\{1, \sqrt{E(s)/\beta}\}$. If $\eta < 1$, then $4c_B m T \eta^2 = 4c_B m T E(s)/\beta = \frac{\delta}{2} E(s)$. If $\eta = 1$, then $E(s) \ge \beta$ and $4c_B m T \le \frac{\delta}{2} E(s)$. In both cases the last term of (4.5) is at most $\frac{\delta}{2} h(\eta) E(s)$, and

$$E(s) - E(s+T) \ge \frac{\delta}{2K} h(\eta) E(s) = \rho\, \Psi_\beta(E(s)).$$

This also holds trivially when $E(s) = 0$, and proves (4.2) for $u_0 \in D$.

To prove (ii), let $E_k := E(kT)$ for integers $k \ge 0$. While $E_k > \beta$, we have $\Psi_\beta(E_k) = \kappa_0 E_k$, so $E_{k+1} \le (1-\rho\kappa_0) E_k$; since $(1-\rho\kappa_0)^{\bar{k}_0} E(0) \le \beta$ by the definition of $\bar{k}_0$, the first index $k_0 := \min\{k \ge 0 : E_k \le \beta\}$ satisfies $k_0 \le \bar{k}_0$. For $k \ge k_0$ we have $E_k \le \beta$ by monotonicity, so $e_k := E_k/\beta \in [0,1]$ and $\Psi_\beta(E_k) = \beta M(e_k)$; hence $e_{k+1} \le e_k - \rho M(e_k)$. Since $\rho \le \frac{1}{4}$, Lemma 3.2 applies with $\varrho = \rho$ and gives $e_k \le \Phi^{-1}(\rho(k-k_0))$ for every integer $k \ge k_0$. Now let $t \ge 0$ and let $k := \lfloor t/T \rfloor$ be

the unique integer with $kT \le t < (k+1)T$. Since $E$ is nonincreasing, $E(t) \le E_k$. If $k \ge \bar{k}_0$, then $k \ge k_0$ and $k - k_0 \ge k - \bar{k}_0 \ge 0$; as $\Phi^{-1}$ is nonincreasing,

$$E(t) \le E_k = \beta e_k \le \beta\, \Phi^{-1}\big(\rho(k-k_0)\big) \le \beta\, \Phi^{-1}\big(\rho(k-\bar{k}_0)\big) = \beta\, \Phi^{-1}\big(\rho(\lfloor t/T\rfloor - \bar{k}_0)\big),$$

which is (4.3). Thus the index $k$ in Lemma 3.2 is always the number $\lfloor t/T\rfloor$ of complete time steps of length $T$ contained in $[0,t]$, and the shift $\bar{k}_0$ is applied to this integer.

Finally, for $u_0 \in X$ take $D \ni u_{0,n} \to u_0$. Then $\mathcal{S}(t)u_{0,n} \to \mathcal{S}(t)u_0$ for each $t$, so the monotonicity of $E$ and (4.2) (in which $\Psi_\beta$ is continuous) pass to the limit; (ii) is a consequence of (i).

(iii) If $h \ge \kappa_*$, then $\Psi_\beta(e) \ge \kappa_* e$ and (4.2) gives $E(s+T) \le (1-\rho\kappa_*)E(s)$. If (P3) holds for all $|f|$, the last statement of Lemma 3.1 gives (4.5) with $h(\eta)$ replaced by 1 and without the last term, i.e. $E(s+T) \le (1-\delta/K)E(s)$. □

*Remark* 4.7. In [7] the constants depend on $E(0)$ through the weight function. Here the only dependence is the shift $\bar{k}_0$, which is logarithmic in $E(0)$: after a waiting time $\bar{k}_0 T$, during which the decay is exponential, the bound (4.3) is universal. Constants depend on $T$ only through $\beta$ and the time unit.

### 4.1. **Explicit rates.**

**Corollary 4.8.** *If* $\inf_{0<s\le 1} \min\{g_1(s)/s, g_2(s)/s\} =: \kappa_* > 0$, *then* $E(t) \le (1-\rho\kappa_*)^{\lfloor t/T\rfloor} E(0)$ *for all data and all* $t \ge 0$.

*Proof.* $h \ge \kappa_*$; apply Theorem 4.6(iii). □

**Corollary 4.9.** *If* $g_i(s) \ge c\, s^p$ *on* $[0,1]$ *for some* $p > 1$, $c \in (0,1]$, $i = 1, 2$, *then for* $\lfloor t/T\rfloor \ge \bar{k}_0$

$$E(t) \le \beta\Big(1 + \frac{c(p-1)\rho}{2}\big(\lfloor t/T\rfloor - \bar{k}_0\big)\Big)^{-\frac{2}{p-1}}.$$

*In particular* $E(t) \le C(1+t)^{-2/(p-1)}$, *with* $C$ *independent of the data if* $E(0) \le \beta$. *No monotonicity of* $g_i(s)/s$, *no convexity and no regularity of* $g_i$ *are needed.*

*Proof.* $\ell_i(s) \ge c s^{p-1}$, so $h(\sigma) \ge c\sigma^{p-1}$ and, by (2.3), $\Phi(x) \le \frac{2}{c}\int_{\sqrt{x}}^1 \sigma^{-p}\,\mathrm{d}\sigma \le \frac{2}{c(p-1)}\big(x^{-\frac{p-1}{2}} - 1\big)$. Hence $\Phi^{-1}(\tau) \le (1 + \frac{c(p-1)}{2}\tau)^{-2/(p-1)}$. □

**Example 4.10.** Let $g(s) = \frac{1}{4}s^2\big(3 + \sin(5\ln\frac{1}{s})\big)$ for $0 < s \le 1$, $g(0) = 0$. Then $g$ is strictly increasing, $\frac{1}{2}s^2 \le g(s) \le s^2 \le s$, $g(s)/s$ is not monotone on any interval $(0,\eta)$, and $K(x) = \sqrt{x}\, g(\sqrt{x})$ is not convex on any interval $(0, x_0)$: with $\varphi = 5\ln\frac{1}{s}$, $g'(s) = \frac{s}{4}(6 + 2\sin\varphi - 5\cos\varphi) \ge \frac{s}{4}(6 - \sqrt{29}) > 0$; $(g(s)/s)' = \frac{1}{4}(3 + \sin\varphi - 5\cos\varphi)$ takes both signs on every $(0,\eta)$; and with $\lambda = \ln\frac{1}{x}$, $4\sqrt{x}\,K''(x) = \frac{9}{4} - \frac{11}{2}\sin\frac{5\lambda}{2} - 5\cos\frac{5\lambda}{2}$, whose minimum $\frac{9}{4} - \frac{\sqrt{221}}{2}$ is negative and is attained along a sequence $x \to 0$. For a feedback $\varrho$ with $\frac{1}{2}g(|f|) \le |\varrho(f)| \le 2g(|f|)$ near 0, (P2) holds with $g_1 = g$, $g_2(s) = s$, $c_1 = \frac{1}{2}$ and $c_2 = 2$ (because $|r| \le 2g(|f|) \le 2|f|$); since $g(s) \ge \frac{1}{2}s^2$, Corollary 4.9 with $p = 2$ and $c = \frac{1}{2}$ gives $E(t) \le C(1+t)^{-2}$ (Figure 2). The same feedback is also covered by convexity methods once $g$

is replaced by the minorant $s^2/2$; the point is that no preprocessing is needed here, and Proposition 4.12 shows that such preprocessing never gains more than constants.

**Corollary 4.11.** (a) *If* $g_i(s) \geq s/(1+\ln\frac{1}{s})$ *on* $(0,1]$, *then for* $\lfloor t/T\rfloor \geq \bar{k}_0$

$$E(t) \leq \beta e^2 \exp\Big(-2\sqrt{1+\rho(\lfloor t/T\rfloor - \bar{k}_0)}\Big),$$

*a stretched exponential* $e^{-c\sqrt{t}}$. (b) *If* $g_i(s) \geq e^{-1/s}$ *on* $(0,1]$, *then* $E(t) \leq C/(\ln t)^2$ *for* $t$ *large.*

*Proof.* (a) Here $h(\sigma) \geq 1/(1+\ln\frac{1}{\sigma})$, because this function is nondecreasing. With $L = \ln\frac{1}{\sqrt{x}}$, (2.3) gives $\Phi(x) \leq 2\int_{\sqrt{x}}^1 (1+\ln\frac{1}{\sigma})\frac{\mathrm{d}\sigma}{\sigma} = 2L + L^2$. Hence $\Phi(x) = \tau$ forces $L \geq \sqrt{1+\tau} - 1$, i.e. $x \leq e^{2-2\sqrt{1+\tau}}$. (b) $s \mapsto e^{-1/s}/s$ is nondecreasing on $(0,1]$, so $h(\sigma) \geq e^{-1/\sigma}/\sigma$ and $\Phi(x) \leq 2\int_{\sqrt{x}}^1 e^{1/\sigma}\,\mathrm{d}\sigma \leq 2e^{1/\sqrt{x}}$. Thus $\Phi(x) = \tau$ forces $1/\sqrt{x} \geq \ln(\tau/2)$, i.e. $x \leq (\ln(\tau/2))^{-2}$. □

These rates are those of the amplitude equation $z' = -c\,g(z)$; for the one-dimensional wave equation with boundary feedback they are known to be optimal for the polynomial and logarithmic classes [27]. For the near-linear family $s(1+\ln\frac{1}{s})^{-\alpha}$, $\alpha > 0$, which contains the profile of (a), the stretched exponential rate is shown to be optimal for localized damping in [1]. Figure 4 displays these bounds side by side.

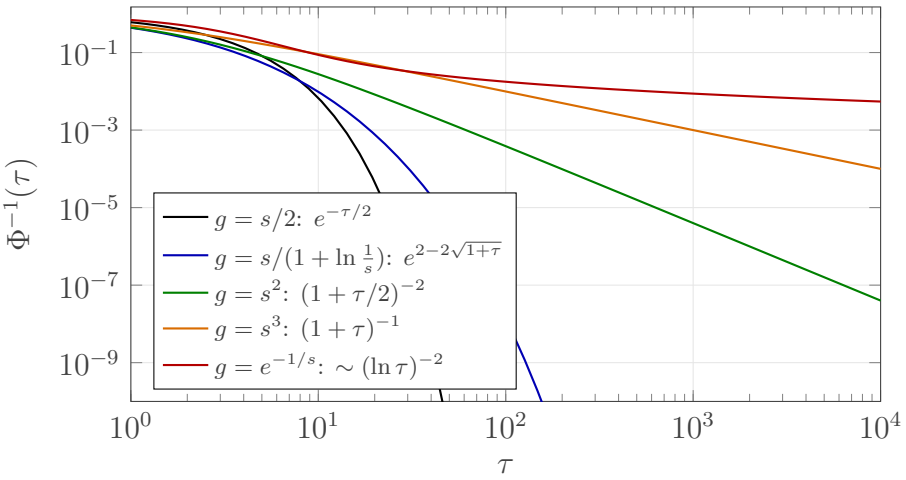



FIGURE 4. The normalized bound $E/\beta \leq \Phi^{-1}(\tau)$ of Theorem 4.6, with $\tau = \rho(\lfloor t/T\rfloor - \bar{k}_0)$, for five profiles $g_1 = g_2 = g$, computed exactly from (2.3). The five classes of Corollaries 4.8–4.11 appear: exponential, stretched exponential, polynomial and logarithmic. For the near-linear profile (blue), $\Lambda_H(x) \to 1$ as $x \to 0$, so the simplified rate $(H')^{-1}(c/t)$ of [7, Theorem 1] is not available, while (4.3) applies directly.

4.2. **Never worse than convexification.** The rates of [7, 4] are expressed through $H(x) = \sqrt{x}\, g(\sqrt{x})$, assumed convex, and the index $\Lambda_H(x) = H(x)/(xH'(x))$. Under $\Lambda_H \le \lambda < 1$, [7, Theorem 1] gives $E(t) \lesssim (H')^{-1}(c/t)$.

**Proposition 4.12.** *Let $g_1 = g_2 = g$ satisfy (g), let $H(x) = \sqrt{x}\, g(\sqrt{x})$ be convex and $C^1$ on $[0,1]$, and assume $\Lambda_H(x) \le \lambda < 1$ on $(0,1]$. Let $(H')^{-1}(y) := \sup\{x \in [0, \frac{1}{2}] : H'(x) \le y\}$. Then the bound (4.3) implies*

$$E(t) \le 2\beta\, (H')^{-1}\Big(\frac{2\lambda}{(1-\lambda)\rho\,(\lfloor t/T\rfloor - \bar{k}_0)}\Big) \qquad (\lfloor t/T\rfloor > \bar{k}_0).$$

*Proof.* Since $H$ is convex with $H(0) = 0$, $\ell(\sigma) = g(\sigma)/\sigma = H(\sigma^2)/\sigma^2$ is nondecreasing, so $h = \ell$ and $\Phi(x) = 2\Theta(\sqrt{x})$ by (2.3). Let $\kappa := \frac{1}{\lambda} - 1 > 0$. The condition $\Lambda_H \le \lambda$ reads $xH'(x) \ge (1+\kappa)H(x)$, i.e. $x \mapsto H(x)/x^{1+\kappa}$ is nondecreasing; equivalently $\sigma \mapsto \ell(\sigma)/\sigma^{2\kappa}$ is nondecreasing. Hence, for $0 < z \le \sigma \le 1$, $\ell(\sigma) \ge \ell(z)(\sigma/z)^{2\kappa}$, and

$$\Theta(z) = \int_z^1 \frac{\mathrm{d}\sigma}{\sigma\ell(\sigma)} \le \frac{z^{2\kappa}}{\ell(z)} \int_z^\infty \sigma^{-1-2\kappa}\, \mathrm{d}\sigma = \frac{1}{2\kappa\, \ell(z)}.$$

Let $\tau := \rho(\lfloor t/T\rfloor - \bar{k}_0) > 0$ and $z := \sqrt{\Phi^{-1}(\tau)} = \Theta^{-1}(\tau/2) \in (0,1]$. Then $\ell(z) \le 1/(2\kappa\Theta(z)) = 1/(\kappa\tau)$. Let $x := z^2/2 \le \frac{1}{2}$. By convexity, $H(2x) \ge H(x) + xH'(x) \ge xH'(x)$, so

$$H'(x) \le \frac{H(z^2)}{x} = 2\ell(z) \le \frac{2}{\kappa\tau} = \frac{2\lambda}{(1-\lambda)\tau}.$$

Hence $x \le (H')^{-1}(2\lambda/((1-\lambda)\tau))$, and $E(t) \le \beta z^2 = 2\beta x$ by (4.3). □

The bound of Proposition 4.12 has the same form as [7, Theorem 1]. When $\Lambda_H(x) \to 1$ (feedbacks "close to linear"), the simplified rate $(H')^{-1}(c/t)$ is not available, while (4.3) still applies and gives, e.g., the stretched exponential of Corollary 4.11(a).

For profiles with $\Lambda_H \to 1$ the comparison is made at the level of dissipation functions. Convexification methods replace $K(x) := x \min_i \ell_i(\sqrt{x})$ by its largest convex minorant $K^{**}$ on $[0,1]$ and iterate (or integrate) with $K^{**}$; see [17, 3, 4].

**Proposition 4.13.** *Let $K(x) := x \min_i \ell_i(\sqrt{x})$ for $0 < x \le 1$, $K(0) = 0$, and let $K^{**}$ be its largest convex minorant on $[0,1]$. Then $K^{**} \le M$ on $[0,1]$, where $M(y) = y\, h(\sqrt{y})$ is the function of (2.2). Consequently, for every $\varrho \in (0,1]$, the bound of Lemma 3.2 is at least as good as the bound obtained by iterating $e_{k+1} \le e_k - \varrho K^{**}(e_k)$.*

*Proof.* Since 0 is a convex minorant of $K$, $K^{**} \ge 0$ and $K^{**}(0) = 0$. For $0 < y \le y' \le 1$, convexity gives $K^{**}(y) \le \frac{y}{y'} K^{**}(y') \le \frac{y}{y'} K(y') = y \min_i \ell_i(\sqrt{y'})$. Taking the infimum over $y' \in [y,1]$ gives $K^{**}(y) \le y \min_i \min_{s\in[\sqrt{y},1]} \ell_i(s) = y\, h(\sqrt{y}) = M(y)$. For the last claim, put $\Phi_{**}(x) := \int_x^1 \mathrm{d}y/K^{**}(y)$ (with $\Phi_{**} = +\infty$ if $K^{**}$ vanishes on $(0,x]$); then $\Phi \le \Phi_{**}$, hence $\Phi^{-1} \le \Phi_{**}^{-1}$ wherever the latter is defined. □

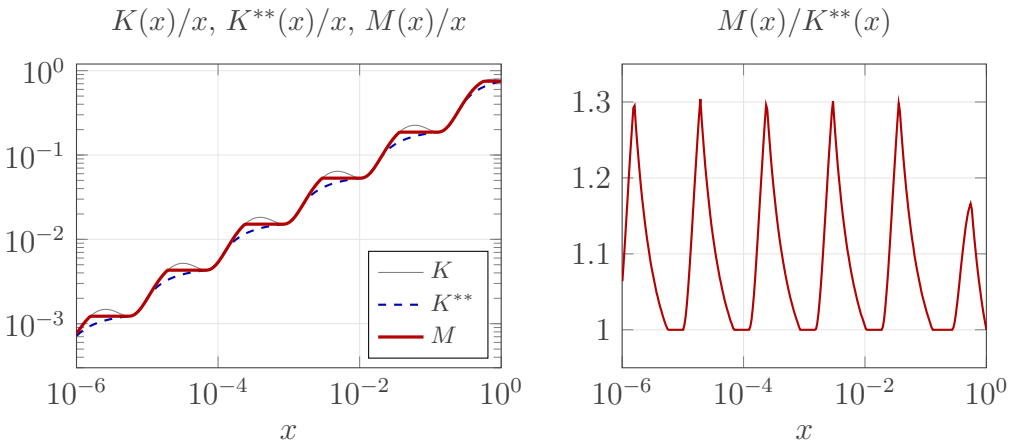



FIGURE 5. Proposition 4.13 for the oscillating profile $g(s) = \frac{1}{4}s^2(3+\sin(5\ln\frac{1}{s}))$ of Example 4.10. Left, divided by $x$ for legibility: the function $K(x) = x\,\ell(\sqrt{x}) = \sqrt{x}\,g(\sqrt{x})$, which is not convex near 0; its largest convex minorant $K^{**}$, which drives convexification methods; and the dissipation function $M(x) = x\,h(\sqrt{x})$ of Theorem 4.6. Right: $M \geq K^{**}$ everywhere, with a gain of up to 30%; the envelope never loses anything with respect to convexification and requires no preprocessing of $g$.

4.3. **Relation with observability.** The next proposition shows that the hypothesis of [7, Theorem 1] implies (L), with an explicit $\delta$. With $B$ replaced by $B+D$ it does the same for [7, Theorem 2], where $D = (\Delta x)^\sigma V_{\Delta x}$ is uniformly bounded by [7, (32)]. The converse, (L) $\Rightarrow$ observability, is classical [16].

**Proposition 4.14.** *Let $\mathcal{A}$ be skew-adjoint and $B \in \mathcal{L}(X)$ self-adjoint and nonnegative. Assume that for some $T > 0$ and $C_T > 0$*

$$C_T\|\varphi(0)\|^2 \leq \int_0^T \|B^{1/2}\varphi(t)\|^2\,\mathrm{d}t \qquad \text{for every solution of } \varphi' + \mathcal{A}\varphi = 0.$$

*Then every solution of $z' + \mathcal{A}z + Bz = 0$ satisfies $\|z(T)\|^2 \leq (1-\delta)\|z(0)\|^2$ with* $\delta = \dfrac{2C_T}{2 + T^2\|B\|^2}$.

*Proof.* Let $\varphi(0) = z(0) \in D(\mathcal{A})$ (then argue by density) and $\theta = \varphi - z$, so that $\theta' + \mathcal{A}\theta = Bz$ and $\theta(0) = 0$. Since $e^{-t\mathcal{A}}$ is unitary, $\|\theta(t)\| \leq \int_0^t \|Bz\| \leq \sqrt{t}(\int_0^t \|Bz\|^2)^{1/2}$, so $\int_0^T \|\theta\|^2 \leq \frac{T^2}{2}\int_0^T \|Bz\|^2 \leq \frac{T^2\|B\|}{2}\int_0^T \|B^{1/2}z\|^2$. Therefore

$$\int_0^T \|B^{1/2}\varphi\|^2 \leq 2\int_0^T \|B^{1/2}z\|^2 + 2\|B\|\int_0^T \|\theta\|^2 \leq (2 + T^2\|B\|^2)\int_0^T \|B^{1/2}z\|^2,$$

and $\int_0^T \|B^{1/2}z\|^2 = \frac{1}{2}(\|z(0)\|^2 - \|z(T)\|^2)$. □

The constant $2 + T^2\|B\|^2$ improves the constant $8T^2\|B^{1/2}\|^4 + 2$ of [7, Lemma 2]. The point of Theorem 4.6, however, is that no such constant enters when (L) is available directly.

## 5. Discrete time: the midpoint rule with viscosity

Let $\Delta t > 0$ and let $V$ be a nonnegative self-adjoint operator on $X$, possibly unbounded; for instance $V = (\Delta t)^2 A^*A$ [14]. Following [14, 7], consider

$$\begin{cases} \dfrac{\tilde{u}^{k+1} - u^k}{\Delta t} + Am^k + C^*G(m^k) = 0, \qquad m^k := \dfrac{u^k + \tilde{u}^{k+1}}{2}, \\ \tilde{u}^{k+1} = (I + \Delta t\, V)u^{k+1}, \end{cases} \tag{5.1}$$

with $m^k \in D(A)$ and $u^{k+1} \in D(V)$. The linear damped scheme is (5.1) with $C^*G$ replaced by $C^*C$. Write

$$\nu(x) := \langle Vx, x\rangle + \frac{\Delta t}{2}\|Vx\|^2 \qquad (x \in D(V)),$$

a nonnegative quadratic form, so that $\nu(x+y) \le 2\nu(x) + 2\nu(y)$.

**Lemma 5.1.** *Every solution of* (5.1) *satisfies*

$$\tfrac{1}{2}\|u^{k+1}\|^2 - \tfrac{1}{2}\|u^k\|^2 = -\Delta t\big(\nu(u^{k+1}) + q(m^k) + \mathrm{Re}\langle G(m^k), Cm^k\rangle\big).$$

*Proof.* Take the real part of the inner product of the first equation with $m^k$. Since $\mathrm{Re}\langle \tilde{u} - u^k, (\tilde{u} + u^k)/2\rangle = \frac{1}{2}(\|\tilde{u}\|^2 - \|u^k\|^2)$, we get $\frac{1}{2}\|\tilde{u}^{k+1}\|^2 - \frac{1}{2}\|u^k\|^2 = -\Delta t(q(m^k) + \mathrm{Re}\langle G(m^k), Cm^k\rangle)$. The second equation gives $\|\tilde{u}^{k+1}\|^2 = \|u^{k+1}\|^2 + 2\Delta t\, \nu(u^{k+1})$. □

**Proposition 5.2.** *Assume* (A), (g), (P), *let* $u^k \in X$ *and* $\mathcal{N} = C^*G$. *Then:* (a) *every solution* $m^k$ *of the first equation of* (5.1) *satisfies* $\|m^k\| \le \|u^k\|$; (b) *if* $\dim X < \infty$, *a solution exists for every* $\Delta t > 0$; (c) *if* $\mathcal{N}$ *is monotone, the solution exists and is unique for every* $\Delta t > 0$; (d) *if* $\mathcal{N}$ *is Lipschitz with constant* $L_R$ *on the ball of radius* $R = 2\|u^k\|$ *and* $\Delta t L_R \le 1$, *the solution exists and is unique. In all cases* $u^{k+1} = (I + \Delta t V)^{-1}(2m^k - u^k)$ *is then well defined.*

*Proof.* The first equation is equivalent to $m + \frac{\Delta t}{2}(Am + \mathcal{N}(m)) = u^k$. (a) Take the real inner product with $m$ and use $q \ge 0$, $\mathrm{Re}\langle \mathcal{N}(m), m\rangle \ge 0$: $\|m\|^2 \le \mathrm{Re}\langle u^k, m\rangle \le \|u^k\|\|m\|$. (b) Let $\Psi(m)$ be the left-hand side minus $u^k$, on the underlying real space with inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$. Then $\mathrm{Re}\langle \Psi(m), m\rangle \ge \|m\|^2 - \|u^k\|\|m\| > 0$ on the sphere $\|m\| = \|u^k\| + 1$, and the acute-angle lemma, a consequence of Brouwer's theorem [18, Chapter 1, Section 4], gives a zero. (c) $A + \mathcal{N}$ is maximal monotone (proof of Proposition 4.3), so $I + \frac{\Delta t}{2}(A + \mathcal{N})$ is a bijection of $D(A)$ onto $X$ [12, Chapter II]. (d) Let $\mathcal{R} = (I + \frac{\Delta t}{2}A)^{-1}$, so that $\|\mathcal{R}\| \le 1$. The map $\Gamma(m) = \mathcal{R}(u^k - \frac{\Delta t}{2}\mathcal{N}(m))$ sends the ball $\bar{B}_R$ into itself, because $\mathcal{N}(0) = 0$, and is a contraction there; by (a) every solution lies in $\bar{B}_R$. (The same contraction principle, with explicit error estimates, underlies the operator-theoretic convergence analysis of decomposition methods in [22].) □

**Hypothesis 5.3** ($\mathrm{L}_{\Delta t}$)**.** There exist an integer $N \ge 1$ and $\delta \in (0, 1]$ such that every solution of the linear damped scheme satisfies $\|z^N\|^2 \le (1 - \delta)\|z^0\|^2$. We set $T_N := N\Delta t$.

**Lemma 5.4.** *Assume* (A), (P0), (P1). *Let* $(u^k)$ *solve* (5.1) *and let* $(z^k)_{j\le k\le j+N}$ *solve the linear damped scheme with* $z^j = u^j$*; write* $m_u^k$, $m_z^k$ *for the midpoints. Then*

$$\Delta t \sum_{k=j}^{j+N-1} \big(\nu(z^{k+1}) + q(m_z^k) + \|Cm_z^k\|^2\big) \le 2\Delta t \sum_{k=j}^{j+N-1} \big(\nu(u^{k+1}) + q(m_u^k)\big) + 4\Delta t \sum_{k=j}^{j+N-1} \big(\|Cm_u^k\|^2 + \|G(m_u^k)\|^2\big).$$

*Proof.* Take $j = 0$ and let $\psi^k = u^k - z^k$, $\tilde\psi^k = \tilde u^k - \tilde z^k$, $m_\psi^k = m_u^k - m_z^k$. Subtracting the schemes,

$$\frac{\tilde\psi^{k+1} - \psi^k}{\Delta t} + Am_\psi^k + C^*\big(G(m_u^k) - Cm_z^k\big) = 0, \qquad \tilde\psi^{k+1} = (I + \Delta t V)\psi^{k+1}.$$

As in Lemma 5.1,

$$\frac{\|\psi^{k+1}\|^2 - \|\psi^k\|^2}{2\Delta t} + \nu(\psi^{k+1}) + q(m_\psi^k) + \mathrm{Re}\langle G(m_u^k) - Cm_z^k, Cm_u^k - Cm_z^k\rangle = 0.$$

Exactly as in Lemma 4.5, the last term is $\ge \frac12\|Cm_z^k\|^2 - \|G(m_u^k)\|^2 - \|Cm_u^k\|^2$. Summing over $k$ and using $\psi^0 = 0$,

$$\Delta t \sum_k \big(\nu(\psi^{k+1}) + q(m_\psi^k) + \tfrac12\|Cm_z^k\|^2\big) \le \mathsf{S} := \Delta t \sum_k \big(\|G(m_u^k)\|^2 + \|Cm_u^k\|^2\big).$$

Conclude with $\nu(z^{k+1}) \le 2\nu(u^{k+1}) + 2\nu(\psi^{k+1})$ and $q(m_z^k) \le 2q(m_u^k) + 2q(m_\psi^k)$. □

**Theorem 5.5.** *Assume* (A), (g), (P) *and* ($\mathrm{L}_{\Delta t}$). *Let* $(u^k)_{k\ge0}$ *be any solution of* (5.1) *and* $E_k := \frac12\|u^k\|^2$. *With* $K, \rho$ *as in Theorem 4.6,* $\beta_N := 8c_B m T_N/\delta$, *and* $\bar k_0$ *defined with* $(\beta_N, E_0)$ *in place of* $(\beta, E(0))$, *the sequence* $(E_k)$ *is nonincreasing,*

$$E_{(\ell+1)N} \le E_{\ell N} - \rho\,\Psi_{\beta_N}(E_{\ell N}) \qquad (\ell \ge 0),$$

*and* $E_k \le \beta_N \Phi^{-1}(\rho(\lfloor k/N\rfloor - \bar k_0))$ *whenever* $\lfloor k/N\rfloor \ge \bar k_0$. *Assertion (iii) of Theorem 4.6 holds with* $T$ *replaced by* $T_N$ *and* $t/T$ *by* $k/N$. *The constants depend only on* $\delta, T_N, m, c_0, c_1, c_2$ *and* $g_1, g_2$*; in particular they do not depend on* $\Delta t$, $V$, $A$, $\|C\|$ *or* $X$.

*Proof.* Monotonicity follows from Lemma 5.1. Fix $j = \ell N$ and let $z$ be the linear damped solution with $z^j = u^j$. By ($\mathrm{L}_{\Delta t}$) and Lemma 5.1 applied to $z$ (with $C^*C$ in place of $C^*G$),

$$\delta E_j \le \Delta t \sum_{k=j}^{j+N-1} \big(\nu(z^{k+1}) + q(m_z^k) + \|Cm_z^k\|^2\big).$$

Multiply by $w = h(\eta) \in (0,1]$ and use Lemma 5.4 and Lemma 3.1 with $x = m_u^k$:

$$\delta h(\eta) E_j \le 2\Delta t \sum \big(\nu(u^{k+1}) + q(m_u^k)\big) + 4c_A \Delta t \sum \mathrm{Re}\langle G(m_u^k), Cm_u^k\rangle + 4c_B m T_N \eta^2 h(\eta).$$

By Lemma 5.1, $\delta h(\eta)E_j \le K(E_j - E_{j+N}) + 4c_B m T_N \eta^2 h(\eta)$. The choice of the level and the iteration in the proof of Theorem 4.6 apply verbatim, with $E(s)$ replaced by $E_{\ell N}$. □

**Example 5.6.** Take $X = \mathbb{R}^2$, $A(x,v) = (-v,x)$ (skew-adjoint), $C(x,v) = v$, $V = 0$, and the cubic feedback $G(x,v) = \varrho(v)$ with $\varrho(v) = v^3$ for $|v| \le 1$ and $\varrho(v) = v$ for $|v| > 1$; thus (5.1) is the implicit midpoint rule for $x'' + x + \varrho(x') = 0$. Hypothesis (P) holds with $b = 1$, $m = 1$, $c_0 = c_1 = c_2 = 1$, $\varepsilon_0 = 1$, $g_1(s) = s^3$ and $g_2(s) = s$, so that $h(\eta) = \eta^2$. Since $\varrho$ is monotone, Proposition 5.2(c) gives a unique solution for every $\Delta t > 0$. The linear damped scheme is the midpoint rule for $z'' + z + z' = 0$; a computation with its $2 \times 2$ amplification matrix shows that $\|z^N\|^2 \le 0.05\|z^0\|^2$ for $N = \lceil 4/\Delta t \rceil$ and every $\Delta t \in (0,1]$, so ($\mathrm{L}_{\Delta t}$) holds with $\delta = \frac{1}{2}$ and $T_N \le 5$. Theorem 5.5 and Corollary 4.9 (with $p = 3$) give $E_k \le C(1 + k\Delta t)^{-1}$ with $C$ independent of $\Delta t \in (0,1]$. Figure 6 confirms this: $k\Delta t\, E_k$ stays below 1.05 for $\Delta t = 1, \frac{1}{4}, \frac{1}{20}$, and the exponent $-1$ is attained, in accordance with Proposition 6.9.

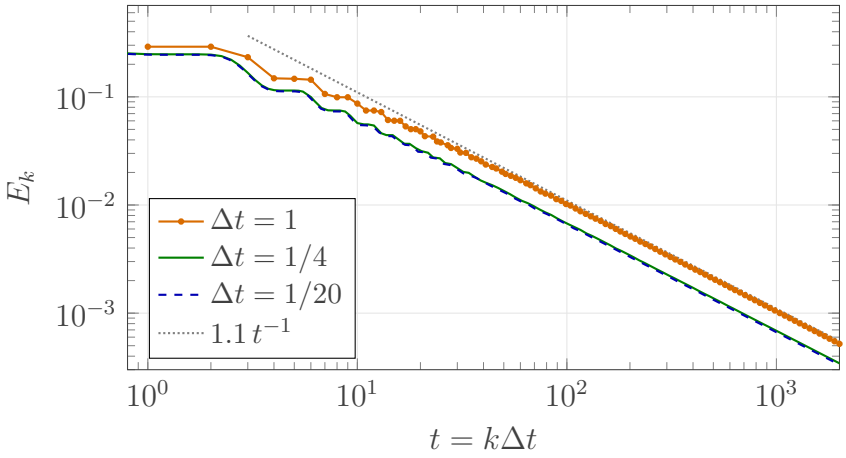



FIGURE 6. Example 5.6: implicit midpoint rule for $x'' + x + (x')^3 = 0$, $x(0) = 0$, $x'(0) = 1$, with three time steps. All discrete energies lie below the common curve $1.1\, t^{-1}$, as predicted by Theorem 5.5, uniformly in $\Delta t$.

**Proposition 5.7.** *Let $\mathcal{A}$ be skew-adjoint, $B \in \mathcal{L}(X)$ self-adjoint and nonnegative, $V \ge 0$ self-adjoint, $N \ge 1$ and $T_N = N\Delta t$. Assume the uniform observability inequality* [7, (44)]*: every solution of the conservative scheme ($B = 0$, same $V$) satisfies*

$$C_T \|\varphi^0\|^2 \le \Delta t \sum_{k<N} \|B^{1/2} m_\varphi^k\|^2 + \Delta t \sum_{k<N} \|V^{1/2} \varphi^{k+1}\|^2 + \Delta t^2 \sum_{k<N} \|V \varphi^{k+1}\|^2.$$

*Then the linear damped scheme with damping $B$ satisfies* ($\mathrm{L}_{\Delta t}$) *with*

$$\delta = \frac{C_T}{2 + \|B\|\big(1 + T_N^2(1 + \|B\|)\big)}.$$

*Proof.* Let $\mathcal{O}(\varphi) = \Delta t \sum_{k<N} \|B^{1/2} m_\varphi^k\|^2 + 2\Delta t \sum_{k<N} \nu(\varphi^{k+1})$. It dominates the right-hand side of the observability inequality, it is a sum of squared seminorms of linear functions of the solution, and $\mathcal{O}(\varphi) \le 2\mathcal{O}(z) + 2\mathcal{O}(\theta)$ for $\theta = \varphi - z$, where $z$ solves the damped scheme with $z^0 = \varphi^0$.

We first bound $\mathcal{O}(z)$. By Lemma 5.1, $\|z^0\|^2 - \|z^N\|^2 = 2\Delta t \sum (\nu(z^{k+1}) + \|B^{1/2} m_z^k\|^2) \ge \mathcal{O}(z)$.

Next we bound $\mathcal{O}(\theta)$. The difference satisfies $(\tilde\theta^{k+1} - \theta^k)/\Delta t + \mathcal{A} m_\theta^k = B m_z^k$ and $\tilde\theta^{k+1} = (I + \Delta t V)\theta^{k+1}$, with $\theta^0 = 0$. Writing $\tilde\theta^{k+1} = (I + \frac{\Delta t}{2}\mathcal{A})^{-1}(I - \frac{\Delta t}{2}\mathcal{A})\theta^k + \Delta t (I + \frac{\Delta t}{2}\mathcal{A})^{-1} B m_z^k$, where the Cayley transform is unitary and the resolvent is a contraction, and using that $(I + \Delta t V)^{-1}$ is a contraction, we get with $a_k := \|B m_z^k\|$: $\|\theta^k\| \le \Delta t \sum_{i<k} a_i$ and $\|m_\theta^k\| \le \Delta t \sum_{i \le k} a_i$. By Cauchy–Schwarz, $\Delta t \sum_{k<N} \|m_\theta^k\|^2 \le \Delta t^3 \frac{N(N+1)}{2} \sum_i a_i^2 \le T_N^2 \Delta t \sum_i a_i^2$. The energy identity for $\theta$, $\|\theta^{k+1}\|^2 + 2\Delta t \nu(\theta^{k+1}) = \|\theta^k\|^2 + 2\Delta t \operatorname{Re}\langle B m_z^k, m_\theta^k\rangle$, gives $2\Delta t \sum \nu(\theta^{k+1}) \le \Delta t \sum a_k^2 + \Delta t \sum \|m_\theta^k\|^2$. Hence $\mathcal{O}(\theta) \le (1 + T_N^2(1 + \|B\|))\Delta t \sum a_k^2$, and $\Delta t \sum a_k^2 \le \|B\| \Delta t \sum \|B^{1/2} m_z^k\|^2 \le \frac{\|B\|}{2}(\|z^0\|^2 - \|z^N\|^2)$.

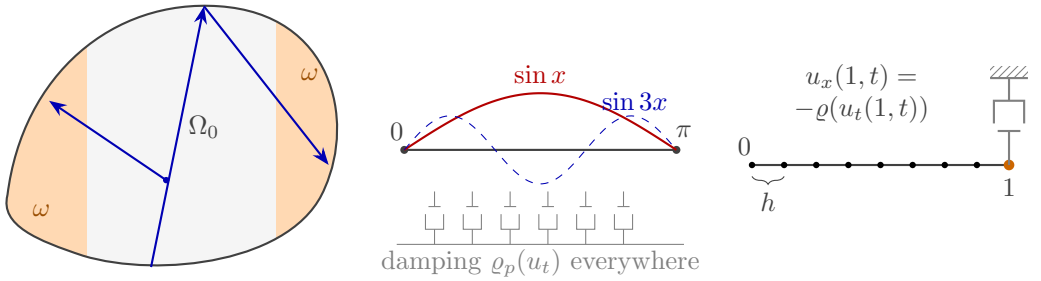
Combining the two bounds, $C_T \|z^0\|^2 \le \big(2 + \|B\|(1 + T_N^2(1 + \|B\|))\big)\big(\|z^0\|^2 - \|z^N\|^2\big)$. □

*Remark* 5.8. The analogous step in [7, Lemma 10] sums the inner products $\langle B m_z, m_\theta\rangle$, which have no sign, against the nonnegative weights $(N - 1 - k)\Delta t$ and bounds the result by $T \sum \langle B m_z, m_\theta\rangle$; this requires taking absolute values first. The proof above works with norms throughout.

## 6. Applications

6.1. **Continuous PDE models.** Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$ boundary, $X = H_0^1(\Omega_0) \times L^2(\Omega_0)$, and $A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}$, which is skew-adjoint. Let $b \in L^\infty(\Omega_0)$, $b \ge 0$, and set $C(u, v) = \sqrt{b}\, v$, $G(u, v) = \sqrt{b}\, \varrho(\cdot, v)$, where $\varrho(x, \cdot)$ is continuous and nondecreasing and satisfies (P2)–(P3) uniformly in $x$. Then $C^* G(u, v) = (0, b\varrho(\cdot, v))$, (P) holds with $f = v$, $r = \varrho(\cdot, v)$, $c_0 = 1$, $m = \|b\|_{L^1(\Omega_0)}$, and (W) follows from Proposition 4.3(a) without any Lipschitz assumption. If $b \ge \alpha > 0$ on an open set $\omega$ such that $(\omega, T)$ satisfies the geometric control condition, the linear locally damped wave equation is exponentially stable; this follows from the observability results of [11], combined with the equivalence of [16], i.e. (L) holds. Theorem 4.6 then yields the rates of [6, 7], now for monotone non-Lipschitz feedbacks, two different profiles above and below, and profiles without convexity or monotonicity of $g(s)/s$ (Example 4.10). The plate, Schrödinger and transport examples of [7, Section 2.2] are treated in the same way; for the Schrödinger feedback $r = f\varrho(x, |f|)$, condition (P) is imposed on $s \mapsto s\varrho(x, s)$. The three geometric situations covered in this section are sketched in Figure 7.

6.2. **Concrete examples.**




(a) $b \geq \alpha > 0$ on $\omega$; every ray of length $T$ meets $\omega$ (GCC)

(b) $u_{tt} - u_{xx} + \varrho_p(u_t) = 0$, $b \equiv 1$, $-\partial_{xx} \geq 1$ (Ex. 7.2)

(c) $N$ cells; boundary node $\|\tau_h\| \sim h^{-1/2}$ (Cor. 6.8)

FIGURE 7. The three geometric settings of Section 6. (a) Localized damping in a domain $\Omega_0$: the damping region $\omega$ (shaded) satisfies the geometric control condition, so the linear damped wave equation satisfies (L). (b) The globally damped string of Example 7.2, for which (L) holds with the explicit constants $T = \ln 6$, $\delta = \frac{1}{2}$ of Proposition 7.1; the first two Galerkin modes are shown. (c) Finite-difference grid with a nonlinear dashpot at the boundary node: the discrete observation grows like $h^{-1/2}$, yet the decay rate of Corollary 6.8 is mesh-independent.

**Example 6.1.** In the setting of the previous subsection, let $0 < \gamma \leq 1 \leq \bar{p}$, let $p : \Omega_0 \to [\gamma, \bar{p}]$ and $a : \Omega_0 \to [a_0, a_1]$ be measurable, with $0 < a_0 \leq a_1$, and let

$$\varrho(x,v) = \begin{cases} a(x)|v|^{p(x)-1}v, & |v| \leq 1, \\ a(x)\, v, & |v| > 1. \end{cases}$$

The exponent may be sublinear on one part of the damping region (where $\varrho(x,\cdot)$ is not Lipschitz at 0) and superlinear on another, and it may vary continuously or jump. Since $\varrho(x,\cdot)$ is continuous and nondecreasing, (W) holds by Proposition 4.3(a). Hypothesis (P) holds with $c_0 = 1$, $\varepsilon_0 = 1$, $c_1 = \min\{a_0, 1\}$, $c_2 = \max\{a_1, 1\}$ and

$$g_1(s) = s^{\bar{p}}, \qquad g_2(s) = s^{1/\gamma}:$$

indeed $a_0|v|^{\bar{p}} \leq |\varrho| \leq a_1$ for $|v| \leq 1$, and $|\varrho|/c_2 \leq |v|^{p(x)}$ gives $(|\varrho|/c_2)^{1/\gamma} \leq |v|^{p(x)/\gamma} \leq |v|$. Let $q := \max\{\bar{p}, 1/\gamma\}$. If $q = 1$ (that is, $p \equiv 1$), Corollary 4.8 gives exponential decay; if $q > 1$, then $\ell_i(s) \geq s^{q-1}$ and Corollary 4.9 gives, under the geometric control condition,

$$E(t) \leq C(1+t)^{-2/(q-1)}.$$

For instance $p = \frac{1}{2}$ on one half of $\omega$ and $p = 3$ on the other gives $q = 3$ and $E(t) \lesssim t^{-1}$, with constants depending only on $a_0, a_1$, on $\|b\|_{L^1(\Omega_0)}$ and on the linear decay constants. The sublinear part takes this feedback out of the scope of [7], which requires a locally Lipschitz feedback; the variable exponent is irrelevant for (P), which is a pointwise condition.

**Example 6.2.** Let $g(s) = \frac{1}{4}s^2\big(3+\sin(5\ln\frac{1}{s})\big)$ be the profile of Example 4.10 and consider

$$x'' + x + \varrho(x') = 0, \qquad \varrho(v) = \begin{cases} \operatorname{sign}(v)\, g(|v|), & |v| \le 1, \\ \frac{3}{4}v, & |v| > 1. \end{cases}$$

With $X = \mathbb{R}^2$, $A(x,v) = (-v,x)$, $C(x,v) = v$ and $G(x,v) = \varrho(v)$, (L) holds because $z'' + z + z' = 0$ is exponentially stable, $\varrho$ is continuous and increasing, so (W) holds by Proposition 4.3(a), and (P) holds as in Example 4.10. Hence $E(t) \le C(1+t)^{-2}$, although $\sqrt{x}\, g(\sqrt{x})$ is not convex on any interval $(0, x_0)$. This rate is attained: for $x(0) = 0$, $x'(0) = 1$ the numerical solution (fourth-order Runge–Kutta, step $5 \cdot 10^{-3}$) satisfies $2.8 \le t^2E(t) \le 5.9$ for $10 \le t \le 3000$, between the values obtained for the two convex profiles $s^2$ and $\frac{1}{2}s^2$ which bracket $g$ (Figure 8). The oscillation of $g(s)/s$ produces a bounded oscillation of $t^2E(t)$, but no loss in the exponent.

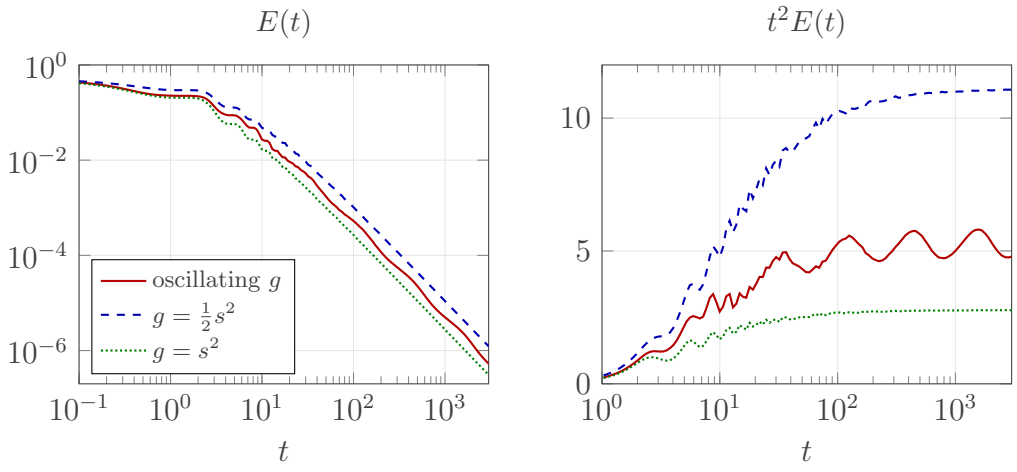



FIGURE 8. Example 6.2: the oscillator $x'' + x + \varrho(x') = 0$ with the oscillating profile of Figure 2, compared with the convex profiles $\frac{1}{2}s^2 \le g \le s^2$. Right: $t^2E(t)$ remains bounded above and below, so the rate $t^{-2}$ given by Theorem 4.6 through the envelope $h(\eta) \ge \eta/2$ is optimal for this feedback.

6.3. **Space semi-discretizations with localized damping: a nodal framework.** Let $(X_h, \langle\cdot,\cdot\rangle_h)$ be finite-dimensional, $\mathcal{A}_h$ skew-adjoint and $Q_h \ge 0$ a numerical viscosity, so $A_h = \mathcal{A}_h + Q_h$. Let $\Omega_h$ be a finite set of nodes carrying weights $\mu_h(\{\omega\}) > 0$ (for finite differences in one dimension, $\mu_h = h\times$counting measure), and $b_h : \Omega_h \to [0,\infty)$. Let $P_h : X_h \to L^2(\Omega_h, \mu_h)$ be linear (the nodal velocity), and set

$$C_h x = \sqrt{b_h}\, P_h x, \qquad G_h(x)(\omega) = \sqrt{b_h(\omega)}\, \varrho\big(\omega, (P_h x)(\omega)\big), \tag{6.1}$$

where $\varrho(\omega,\cdot) : \mathbb{K}^d \to \mathbb{K}^d$ is continuous and satisfies (P1)–(P3) with constants independent of $h$ and $\omega$. Then (P) holds with $m_h = \sum_\omega b_h \mu_h$.

**Corollary 6.3.** *Assume* $\sup_h m_h < \infty$ *and that the linear schemes* $z' + (\mathcal{A}_h + Q_h + C_h^* C_h)z = 0$ *are uniformly exponentially stable:* $\|z(t)\|_h \le M_0 e^{-\nu_0 t}\|z(0)\|_h$ *with* $M_0, \nu_0$ *independent of* $h$. *Then every solution of* $u' + \mathcal{A}_h u + Q_h u + C_h^* G_h(u) = 0$ *satisfies the conclusions of Theorem 4.6 with*

$$T = \frac{\ln(2M_0^2)}{2\nu_0}, \qquad \delta = \frac{1}{2},$$

*and constants independent of* $h$.

*Proof.* With this $T$, $M_0^2 e^{-2\nu_0 T} = \frac{1}{2}$, so (L) holds with $\delta = \frac{1}{2}$. If the solutions are unique (for instance if each $\varrho(\omega, \cdot)$ is monotone, in which case $C_h^* G_h$ is monotone, or locally Lipschitz), Proposition 4.3(c) gives (W) and Theorem 4.6 applies. Otherwise, by the remark following Proposition 4.3, the conclusions of Theorem 4.6 hold for every $C^1$ solution. □

Uniform exponential stability of viscous finite-difference schemes with localized damping is the content of [25] (one dimension and the square) and [21] (two-dimensional domains). Whenever the localized damping in those schemes is discretized nodewise, as in (6.1), Corollary 6.3 gives uniform decay for the corresponding nonlinear schemes. Alternatively, (L) can be derived from the uniform observability hypothesis [7, (36)] through Proposition 4.14, with $B$ replaced by $B_h + (\Delta x)^\sigma V_{\Delta x}$, which is uniformly bounded by [7, (32)].

*Remark* 6.4. Let $X_{\Delta x}$ carry the norm $\|u\|_{\Delta x} = \|\tilde{P}_{\Delta x} u\|_X$, so that $\tilde{P}_{\Delta x}$ is an isometry and $P_{\Delta x} = \tilde{P}_{\Delta x}^*$. Let $B = \mathsf{U} b \mathsf{U}^{-1}$ be the spectral representation used in Remark 2.4. Put $C_{\Delta x} u = \sqrt{b}\, f$ with $f = \mathsf{U}^{-1}\tilde{P}_{\Delta x} u$, and $G_{\Delta x}(u) = \sqrt{b}\, \tilde{\varrho}_{\Delta x}(f)$. Then $C_{\Delta x}^* w = P_{\Delta x}\mathsf{U}(\sqrt{b} w)$, so $C_{\Delta x}^* C_{\Delta x} = P_{\Delta x} B \tilde{P}_{\Delta x} = B_{\Delta x}$, and $\mathsf{U}\tilde{\varrho}_{\Delta x}(f) = \tilde{P}_{\Delta x} P_{\Delta x} F(\tilde{P}_{\Delta x} u)$ gives $C_{\Delta x}^* G_{\Delta x}(u) = B_{\Delta x} F_{\Delta x}(u)$. Hypotheses [7, (34)–(35)] are exactly (P1)–(P3) for this choice, with $c_0 = 1$ and $m \le \|B\|$. Together with Proposition 4.14, this shows that [7, Theorem 2] is a special case of Theorem 4.6. The next remark shows that these hypotheses are restrictive.

*Remark* 6.5. In [7, Section 3.1] the nonlinearity is discretized as $\tilde{\varrho}_{\Delta x}(f) = \mathsf{U}^{-1}\tilde{P}_{\Delta x} P_{\Delta x}\mathsf{U}\varrho(f)$, where $\tilde{P}_{\Delta x} P_{\Delta x}$ is the orthogonal projection onto the approximation space, and sign and growth conditions are required a.e. in $\Omega$ [7, (34)–(35)]. For $P_1$ finite elements these conditions fail. Take a uniform mesh of $[0, 1]$ with 10 cells, $\varrho(s) = s^3$, and $f = \varphi_5$, the hat function at the midpoint. The $L^2$ projection of $f^3$ onto the $P_1$ space has nodal values

$$\ldots,\ 0.0249,\ -0.0928,\ 0.6464,\ -0.0928,\ 0.0249,\ \ldots$$

(Figure 9). Consequently $f\tilde{\varrho}(f) < 0$ on a set of measure about 0.025, which violates the sign condition, and $\tilde{\varrho}(f) \ne 0$ on a set of measure 0.8 where $f = 0$, which violates the upper bound $|\tilde{\varrho}| \le c_2 g^{-1}(|f|)$. The nodal formulation (6.1), which corresponds to finite differences or to lumped-mass finite elements, avoids this difficulty: there (P) is a genuinely nodal condition, inherited from the continuous one.

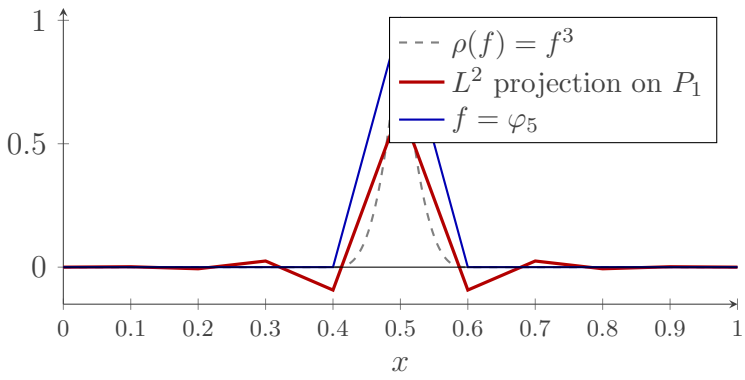



FIGURE 9. Remark 6.5: for $P_1$ finite elements, the orthogonal projection of $\varrho(f) = f^3$ (with $f$ a hat function) oscillates. It has the wrong sign near the support of $f$ and does not vanish where $f$ does, so the pointwise sign and growth hypotheses of [7, (34)–(35)] fail. The nodal formulation (6.1) avoids this.

### 6.4. **Full discretization with localized damping.**

**Corollary 6.6.** *In the setting of Corollary 6.3, assume moreover that* $\kappa := \sup_h \|Q_h + C_h^* C_h\| < \infty$. *Consider the fully discrete scheme* (5.1) *with* $A = \mathcal{A}_h + Q_h$, $C = C_h$, $G = G_h$ *and viscosity* $V = (\Delta t)^2 \mathcal{A}_h^* \mathcal{A}_h$. *Fix* $\Delta t_0 > 0$. *There exist* $N$, $\delta = \frac{1}{2}$ *and* $T_N \le T_0 + \Delta t_0$, *with* $T_0$ *depending only on* $(M_0, \nu_0, \kappa)$, *such that the conclusions of Theorem 5.5 hold with constants independent of* $h$ *and of* $\Delta t \in (0, \Delta t_0]$.

*Proof.* The linear damped scheme is exactly the scheme [14, (1.9)] for $\dot z = (-\mathcal{A}_h)z - B_h^* B_h z$ with $B_h = (Q_h + C_h^* C_h)^{1/2}$, $\|B_h\|^2 \le \kappa$; the viscosity step of [14] reads $\tilde z^{k+1} = (I + \Delta t V) z^{k+1}$ with $V = -(\Delta t)^2 \mathcal{A}_h^2 = (\Delta t)^2 \mathcal{A}_h^* \mathcal{A}_h$. The family belongs to the class $\mathcal{D}(\kappa^{1/2}, M_0^2, 2\nu_0)$ of [14, Definition 3.1], which does not depend on the spaces. By [14, Theorem 1.1] there are $\mu_1, \nu_1 > 0$ depending only on $(\kappa, M_0, \nu_0)$ such that $\|z^k\|^2 \le \mu_1 e^{-\nu_1 k \Delta t} \|z^0\|^2$ for all $h$ and all $\Delta t > 0$. Choose $T_0$ with $\mu_1 e^{-\nu_1 T_0} = \frac{1}{2}$ and $N = \lceil T_0/\Delta t \rceil$; then $(\mathrm{L}_{\Delta t})$ holds with $\delta = \frac{1}{2}$ and $T_N \le T_0 + \Delta t_0$. Existence of solutions follows from Proposition 5.2(b). Apply Theorem 5.5. □

This gives a complete proof of the statement of [7, Theorem 4], under the nodal structure (6.1), for any feedback satisfying (P), monotone or not; no Lipschitz condition is needed in finite dimension.

### 6.5. **Nonlinear boundary feedback after space discretization.**

**Lemma 6.7.** *Let* $X_h$ *be finite-dimensional,* $M_h \in \mathcal{L}(X_h)$ *and* $\tau_h : X_h \to \mathbb{K}$ *linear, and suppose that* $\mathrm{Re}\langle M_h z, z\rangle_h = q_h(z) + |\tau_h z|^2$ *for all* $z$, *where* $q_h \ge 0$ *is a quadratic form. Then* $M_h = \mathcal{A}_h + Q_h + \tau_h^* \tau_h$, *where* $\mathcal{A}_h = \frac{1}{2}(M_h - M_h^*)$ *is skew-adjoint and* $Q_h$ *is self-adjoint with* $\langle Q_h z, z\rangle = q_h(z) \ge 0$.

*Proof.* The Hermitian part $\frac{1}{2}(M_h + M_h^*)$ has quadratic form $q_h + |\tau_h \cdot|^2$. Set $Q_h = \frac{1}{2}(M_h + M_h^*) - \tau_h^* \tau_h$. □

So any linear semi-discrete scheme $z' + M_h z = 0$ whose energy identity reads $E' = -q_h(z) - |\tau_h z|^2$ has the form required by Corollary 6.3. Here $\tau_h z$ is, for instance, $\sqrt{\alpha}$ times the discrete boundary velocity. Its nonlinear counterpart is obtained by replacing $\tau_h^* \tau_h z$ by $\tau_h^* \varrho(\tau_h z)$.

**Corollary 6.8.** *Let $M_h = \mathcal{A}_h + Q_h + \tau_h^* \tau_h$ be as in Lemma 6.7, and assume that $z' + M_h z = 0$ is exponentially stable uniformly in $h$, with constants $(M_0, \nu_0)$. Let $\varrho : \mathbb{K} \to \mathbb{K}$ be continuous and satisfy (P1)–(P3) with $f = \tau_h u$ and $r = \varrho(f)$ (so $d = 1$, $\Omega$ is a point, $b = 1$, $m = 1$). Then all solutions of*

$$u' + \mathcal{A}_h u + Q_h u + \tau_h^* \varrho(\tau_h u) = 0$$

*satisfy the conclusions of Theorem 4.6 with $T = \ln(2M_0^2)/(2\nu_0)$, $\delta = \frac{1}{2}$, and constants independent of $h$, even though $\|\tau_h\| \to \infty$ is allowed.*

*Proof.* Apply Corollary 6.3 with $U = \mathbb{K}$, $C_h = \tau_h$ and $G_h(u) = \varrho(\tau_h u)$. None of the constants involves $\|\tau_h\|$. □

Tcheugoué Tébou and Zuazua [26] constructed a viscous finite-difference scheme for the one-dimensional wave equation with linear boundary feedback whose energy decays exponentially, uniformly in the mesh size, while the scheme without viscosity does not. Whenever such a scheme has an energy identity of the form of Lemma 6.7, Corollary 6.8 yields uniform decay rates for its nonlinear analogue, for arbitrarily weak feedback $\varrho$. For finite-difference schemes $\|\tau_h\|$ is of order $h^{-1/2}$ in the discrete energy norm, so the approach of [7], whose constants grow with $T^2\|B_h\|^2$, does not apply; this case is listed as open in [7, Section 4].

Figure 10 summarizes, for the experiment of the next subsection, which quantities grow with the mesh and which do not.

6.6. **A numerical illustration.** The following experiment is not used in any proof. We discretize $u_{tt} = u_{xx}$ on $(0,1)$, $u(0,t) = 0$, $u_x(1,t) = -\varrho(u_t(1,t))$, with $\varrho(v) = v^3$ for $|v| \le 1$ and $\varrho(v) = v$ for $|v| > 1$, by finite differences on $N$ cells with lumped mass at the boundary node and the numerical viscosity $h^2 \Delta_h u_t$ in the spirit of [25, 26]. The discrete energy satisfies $E' = -\varrho(v_N) v_N - h^2 \sum_j h|(v_j - v_{j-1})/h|^2$, which is of the form of Lemma 6.7 with $|\tau_h u| = |v_N|$, so $\|\tau_h\| \sim (2/h)^{1/2}$. We integrate in time with the classical fourth-order Runge–Kutta method, $\Delta t = h/4$, from $u_0 = 0$, $u_1 = 0.8 \sin(1.5\pi x)$. For comparison, the exact energy of the continuous problem is computed from d'Alembert's formula by iterating the reflection map $a \mapsto a + (\mathrm{id} + \varrho)^{-1}(-2a)$ on the Riemann invariant [27]. Since $g(s) = s^3$, Theorem 4.6 predicts a bound of order $t^{-1}$, uniform in $h$.

Figure 11 shows that all discrete energies stay below the same curve $0.25\, t^{-1}$ for $t \ge 3$ (the maxima of $tE_h(t)$ are 0.12, 0.16, 0.18, 0.20 for $N = 20, 40, 80, 160$, and 0.24 for the continuous problem). Coarse meshes decay faster, because the numerical viscosity damps all modes; as $h \to 0$ the discrete energies approach the

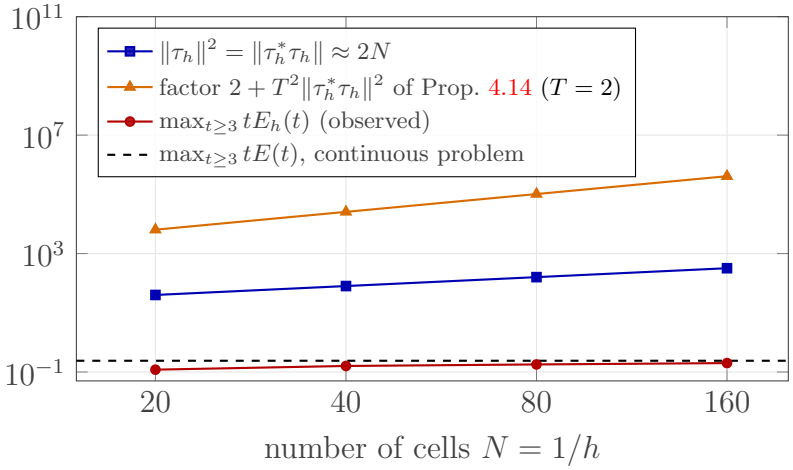



FIGURE 10. Boundary feedback after space discretization (Corollary 6.8 and Figure 11). The discrete boundary observation grows like $h^{-1/2}$, and the comparison with the conservative system (Proposition 4.14, or [7, Lemma 2]) loses a factor that grows like $h^{-2}$ (orange). The constants of Corollary 6.8 involve neither, and the observed quantity $\max_t tE_h(t)$ (red) stays bounded, below the value of the continuous problem.

continuous one from below. This is the behaviour described by Corollary 6.8: an upper bound uniform in $h$, although $\|\tau_h\| \to \infty$.

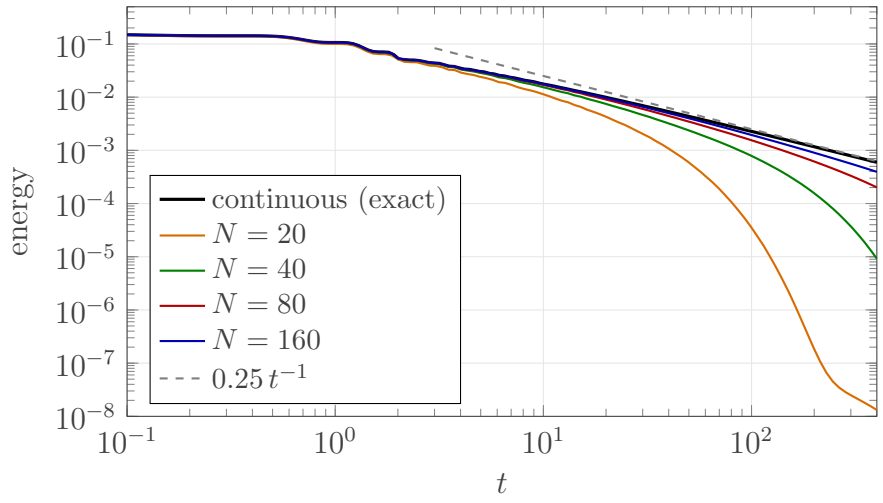



FIGURE 11. Wave equation on $(0, 1)$ with the cubic boundary feedback $u_x(1, t) = -u_t(1, t)^3$, discretized by viscous finite differences with $N$ cells. All discrete energies lie below the common curve $0.25\,t^{-1}$ (dashed), in agreement with the mesh-independent bound of Corollary 6.8, even though the norm of the discrete boundary observation grows like $h^{-1/2}$. The black curve is the exact energy of the continuous problem.

### 6.7. **Sharpness.**

**Proposition 6.9.** *Let $p > 1$. In Theorem 4.6 with $g_1 = g_2 = s^p$, the conclusion is $E(t) \le C(1+t)^{-2/(p-1)}$, and the exponent $2/(p-1)$ cannot be improved in general: there are systems satisfying all hypotheses for which $E(t) \ge c'(1+t)^{-2/(p-1)}$.*

*Proof.* The upper bound is Corollary 4.9. For the lower bound take $X = U = \mathbb{R}$, $A = 0$, $C = 1$ and $G(u) = \varrho(u)$ with $\varrho(s) = |s|^{p-1}s$ for $|s| \le 1$ and $\varrho(s) = s$ for $|s| > 1$. Then (P) holds with $b = 1$, $m = 1$, $c_0 = 1$, $\varepsilon_0 = 1$, $g_1 = g_2 = s^p$ and suitable $c_1, c_2$, and the linear system $z' + z = 0$ satisfies (L) with $\delta = 1 - e^{-2T}$. For $u_0 = 1$, $u(t) = (1+(p-1)t)^{-1/(p-1)}$ and $E(t) = \frac{1}{2}(1+(p-1)t)^{-2/(p-1)}$. The same example can be added as a decoupled component to any of the families above, so the exponent is also optimal uniformly in the discretization parameters. □

For general $g$ with $\limsup \Lambda_H < 1$, optimality of the rate $(H')^{-1}(c/t)$ in finite dimension is proved in [4]; by Proposition 4.12 our bound is at least as good.

## 7. Fully explicit examples and the limits of sharpness

In this section every constant is computed: the linear decay constant $\delta$, the time $T$, the structural constants of (P), the constants $K, \rho, \beta$ of Theorem 4.6 and the resulting bound. We then compare these bounds with accurate numerical solutions. The comparison shows where the theorem is sharp (the exponent), where it is not (the constants), and one situation in which the exponent itself is not attained.

### 7.1. **An explicit linear estimate.**

**Proposition 7.1.** *Let $H$ be a Hilbert space, $\mathcal{A}_0$ a self-adjoint operator on $H$ with $\mathcal{A}_0 \ge \lambda_1 > \frac{1}{4}$, and $X = D(\mathcal{A}_0^{1/2}) \times H$ with $\|(z,w)\|^2 = \|\mathcal{A}_0^{1/2} z\|^2 + \|w\|^2$. Every solution of $z'' + \mathcal{A}_0 z + z' = 0$ satisfies*

$$\|(z,z')(t)\|^2 \le \gamma(\lambda_1)\, e^{-t} \|(z,z')(0)\|^2, \qquad \gamma(\lambda_1) := \frac{2\sqrt{\lambda_1}+1}{2\sqrt{\lambda_1}-1}, \qquad t \ge 0. \quad (7.1)$$

*The constant $\gamma(\lambda_1)$ is optimal when $\lambda_1$ is an eigenvalue of $\mathcal{A}_0$. In particular, if $\mathcal{A}_0 \ge 1$, then (L) holds with $T = \ln 6$ and $\delta = \frac{1}{2}$.*

*Proof.* By the spectral theorem it suffices to consider a scalar $\lambda \ge \lambda_1$ and the variables $Y = (\sqrt{\lambda}\, z, z') \in \mathbb{R}^2$, which satisfy $Y' = -MY$ with $M = \begin{pmatrix} 0 & -k \\ k & 1 \end{pmatrix}$, $k = \sqrt{\lambda}$. Write $M = \frac{1}{2} I + N$ with $N = \begin{pmatrix} -1/2 & -k \\ k & 1/2 \end{pmatrix}$. Then $N^2 = -\omega^2 I$ with $\omega^2 := \lambda - \frac{1}{4} > 0$, hence

$$e^{-Mt} = e^{-t/2}\big(aI + bN\big), \qquad a = \cos \omega t, \quad b = -\frac{\sin \omega t}{\omega}.$$

A direct computation gives $N^{\mathsf T}N = \begin{pmatrix} k^2+\frac14 & k \\ k & k^2+\frac14 \end{pmatrix}$ and $N+N^{\mathsf T} = \operatorname{diag}(-1,1)$, so that $(aI+bN)^{\mathsf T}(aI+bN)$ has largest eigenvalue

$$a^2 + b^2\big(k^2+\tfrac14\big) + \sqrt{a^2b^2+b^4k^2} = \Big(\frac{|b|}{2} + \sqrt{1+\frac{b^2}{4}}\Big)^2,$$

where we used $a^2+b^2\omega^2 = 1$. Thus $\|e^{-Mt}\| = e^{-t/2}\big(\frac{|b|}{2}+\sqrt{1+b^2/4}\big)$, an increasing function of $|b| \le 1/\omega$. At $|b| = 1/\omega$ its square equals $\big(\frac{1}{2\omega}+\frac{\sqrt\lambda}{\omega}\big)^2 = \frac{(\sqrt\lambda+\frac12)^2}{\lambda-\frac14} = \frac{2\sqrt\lambda+1}{2\sqrt\lambda-1}$, which is decreasing in $\lambda$ and is reached at $\omega t = \frac{\pi}{2}$ for $\lambda = \lambda_1$. For $\mathcal{A}_0 \ge 1$ we get $\gamma \le 3$, and $3e^{-\ln 6} = \frac12$. □

### 7.2. **The damped string with a power feedback.**

**Example 7.2.** Let $p \ge 1$ and $\varrho_p(v) = |v|^{p-1}v$ for $|v| \le 1$, $\varrho_p(v) = v$ for $|v| > 1$. Consider

$$u_{tt} - u_{xx} + \varrho_p(u_t) = 0 \quad \text{in } (0,\pi)\times(0,\infty), \qquad u(0,t) = u(\pi,t) = 0, \tag{7.2}$$

with energy $E(t) = \frac12\int_0^\pi (u_x^2+u_t^2)\,\mathrm{d}x$, together with its Galerkin approximations on $\operatorname{span}\{\sin x,\dots,\sin Nx\}$. Take $X = H_0^1\times L^2$ (or its Galerkin subspace), $U = L^2(0,\pi)$ with Lebesgue measure, $b\equiv 1$, $C(u,v) = v$ and $G(u,v) = \varrho_p(v)$; for the Galerkin scheme $C^*$ is the orthogonal projection onto the subspace, so $C^*G$ is the Galerkin projection of the feedback. The constants are:

| quantity | value | reason |
|---|---|---|
| $m$ | $\pi$ | $m = \int_0^\pi b\,\mathrm{d}x$ |
| $c_0,\ \varepsilon_0,\ c_1,\ c_2$ | $1,\ 1,\ 1,\ 1$ | $\varrho_p$ odd and nondecreasing; $\lvert\varrho_p(v)\rvert \le \min\{1,\lvert v\rvert\}$ for $\lvert v\rvert \le 1$ |
| $g_1,\ g_2$ | $s^p,\ s$ | $\lvert\varrho_p(v)\rvert = \lvert v\rvert^p$ and $\lvert\varrho_p(v)\rvert \le \lvert v\rvert$ for $\lvert v\rvert \le 1$ |
| $h(\eta),\ \kappa_0$ | $\eta^{p-1},\ 1$ | $\ell_1 = s^{p-1} \le \ell_2 = 1$, both nondecreasing |
| $T,\ \delta$ | $\ln 6,\ \frac12$ | Proposition 7.1, since $-\partial_{xx} \ge 1$ |
| $c_A,\ c_B,\ K$ | $2,\ 2,\ 8$ | Lemma 3.1, Theorem 4.6 |
| $\rho,\ \beta$ | $\frac{1}{32},\ 32\pi\ln 6 \approx 180.13$ | $\rho = \delta/(2K)$, $\beta = 8c_BmT/\delta$ |

Since $C^*C$ is the identity on velocities, the linear damped Galerkin system is $z'' + \mathcal{A}_N z + z' = 0$ with $\mathcal{A}_N \ge 1$, so Proposition 7.1 gives (L) with the same $T,\delta$ for every $N$. Theorem 4.6 and Corollary 4.9 (with $c=1$) give, for every $N \le \infty$ and all data with $E(0) \le \beta$ (so that $\bar k_0 = 0$),

$$\begin{aligned} E(t) &\le 32\pi\ln 6\,\Big(1+\frac{p-1}{64}\Big\lfloor\frac{t}{\ln 6}\Big\rfloor\Big)^{-\frac{2}{p-1}} && (p>1),\\ E(t) &\le \Big(\frac{31}{32}\Big)^{\lfloor t/\ln 6\rfloor}E(0) && (p=1). \end{aligned} \tag{7.3}$$

For larger data, $\bar{k}_0 = \lceil \ln^+(E(0)/\beta)/\ln\frac{32}{31} \rceil$. The bound involves no constant depending on $N$.

Figure 12 compares (7.3) with the computed energies for $p = 2, 3$, $u(\cdot, 0) = 0$ and two initial velocities: $u_t(\cdot, 0) = \sin x$ (data d1) and $u_t(\cdot, 0) = 0.6 \sin x + 0.5 \sin 3x + 0.3 \sin 5x$ (data d2). We used $N = 16$ and $N = 64$ modes, the feedback evaluated at $4N - 1$ equally spaced nodes (for which the discrete sine transform is exactly orthogonal, so the structure $C^*G$ is preserved with $m \le \pi$), and the classical fourth-order Runge–Kutta method with step $10^{-2}$. The curves for $N = 16$ and $N = 64$ coincide to plotting accuracy, and $t^{2/(p-1)}E(t)$ stays in $[1.36, 1.53]$ for $p = 3$ and in $[5.3, 6.2]$ for $p = 2$ on $50 \le t \le 2000$. The exponent of (7.3) is therefore attained, while its constant is about $7 \cdot 10^3$ (for $p = 3$) to $4 \cdot 10^5$ (for $p = 2$) times too large; see Table 1.

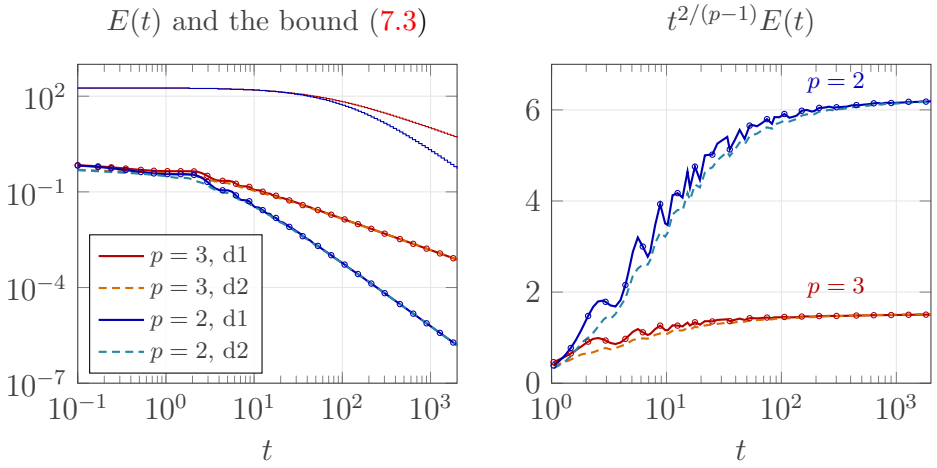



FIGURE 12. Example 7.2: the damped string (7.2). Solid and dashed lines: $N = 16$ modes; circles: $N = 64$ modes (indistinguishable). Thin staircases (left, top): the explicit bound (7.3), valid for every $N$. Right: the compensated energy has a positive plateau, so the exponent $2/(p-1)$ of (7.3) is attained, uniformly in $N$ and for both data.

### 7.3. **Sublinear feedback: the upper profile governs, and the rate is sharp.**

**Example 7.3.** Let $0 < \gamma < 1$ and $x'' + x + \sigma_\gamma(x') = 0$ with $\sigma_\gamma(v) = |v|^{\gamma-1}v$ for $|v| \le 1$ and $\sigma_\gamma(v) = v$ otherwise; $\sigma_\gamma$ is monotone, so (W) holds by Proposition 4.3(a), and $\sigma_\gamma$ is not Lipschitz at 0. With $X = \mathbb{R}^2$, $C(x, v) = v$, $m = 1$, (P) holds with $c_0 = c_1 = c_2 = \varepsilon_0 = 1$, $g_1(s) = s$ and $g_2(s) = s^{1/\gamma}$: for $|v| \le 1$, $|\sigma_\gamma(v)| = |v|^\gamma \ge |v|$ and $|\sigma_\gamma(v)|^{1/\gamma} = |v|$. Thus $h(\eta) = \eta^{1/\gamma - 1}$. Proposition 7.1 with $\lambda_1 = 1$ gives $T = \ln 6$, $\delta = \frac{1}{2}$, and $K = 8$, $\rho = \frac{1}{32}$, $\beta = 32 \ln 6 \approx 57.34$. With $q := 1/\gamma$, Corollary 4.9 gives, for $E(0) \le \beta$,

$$E(t) \le 32 \ln 6 \left(1 + \frac{q-1}{64} \left\lfloor \frac{t}{\ln 6} \right\rfloor\right)^{-\frac{2}{q-1}}. \tag{7.4}$$

Here the decay is limited by the *upper* profile $g_2$: the damping is strong at small velocities, the oscillator becomes overdamped, and the solution creeps towards 0. Balancing $x$ against $\sigma_\gamma(x')$ in the equation gives formally $x' \approx -x^q$, hence $x \approx ((q-1)t)^{-1/(q-1)}$ and

$$E(t) \sim \tfrac{1}{2}\big((q-1)t\big)^{-\frac{2}{q-1}} \qquad (t \to \infty). \tag{7.5}$$

The numerical solution for $x(0) = 0$, $x'(0) = 1$ (Runge–Kutta, step $10^{-3}$) confirms (7.5): $t^2E(t) \in [0.435, 0.499]$ for $\gamma = \frac{1}{2}$ and $tE(t) \in [0.236, 0.250]$ for $\gamma = \frac{1}{3}$ on $30 \le t \le 2000$, with limits $\frac{1}{2}$ and $\frac{1}{4}$ (Figure 13, left). The exponent of (7.4) is therefore attained in the sublinear regime as well. The exponent is determined by $g_2$ alone; the lower profile $g_1(s) = s$ plays no role in it.

### 7.4. **A limit of pointwise methods.**

**Example 7.4.** Let $\epsilon = 0.02$, $s_k = 2^{-k}$ and

$$\varpi(s) = s^2\big(1 - w(s)\big) + s^4 w(s), \qquad w(s) = \exp\Big(-\Big(\frac{s/s_k - 1}{\epsilon}\Big)^2\Big),$$

where $k$ is the integer nearest to $\log_2 \frac{1}{s}$ (in the computation $w$ is set to 0 when $|s/s_k - 1| > 6\epsilon$), so that $\varpi(s) = s^2$ except on the thin intervals $|s/s_k - 1| \lesssim \epsilon$, where it dips to $s^4$. Consider $x'' + x + \varsigma(x') = 0$ with $\varsigma(v) = \operatorname{sign}(v)\varpi(|v|)$ for $|v| \le 1$ and $\varsigma(v) = \varpi(1)v$ otherwise. The feedback is smooth (so (W) holds) but not monotone. (P) holds with $g_1(s) := \min_{[s,1]} \varpi$, $g_2(s) = s$ and $c_0 = c_1 = c_2 = \varepsilon_0 = 1$ (note $\varpi(1) = 1$ and $s^4 \le \varpi(s) \le s^2$). Since $g_1(s) \ge s^4$ and $g_1(s_k) \le s_k^4$, and since $[\eta, 2\eta]$ contains some $s_k$, one has $\eta^3 \le h(\eta) \le 8\eta^3$, and Theorem 4.6 gives exactly the rate $E(t) \lesssim t^{-2/3}$; the convex minorant $K^{**}$ of Proposition 4.13 gives the same rate, because $K^{**}(s_k^2) \le K(s_k^2) \le s_k^5$, so that $K^{**}(x) \le 32x^{5/2}$ by convexity. The numerical solution, however, satisfies $t^2E(t) \in [2.36, 3.20]$ on $20 \le t \le 10^4$, as for the undisturbed profile $s^2$ ($t^2E \in [2.21, 2.77]$); see Figure 13, right. During each oscillation the velocity sweeps the whole interval $[-s, s]$, so the energy lost per period is an average of $\varpi$ over $[0, s]$, in which dips of relative width $\epsilon$ are negligible. Any method in which the feedback is bounded below at each velocity level separately, including ours and the convexity methods of [17, 3, 7], loses this averaging.

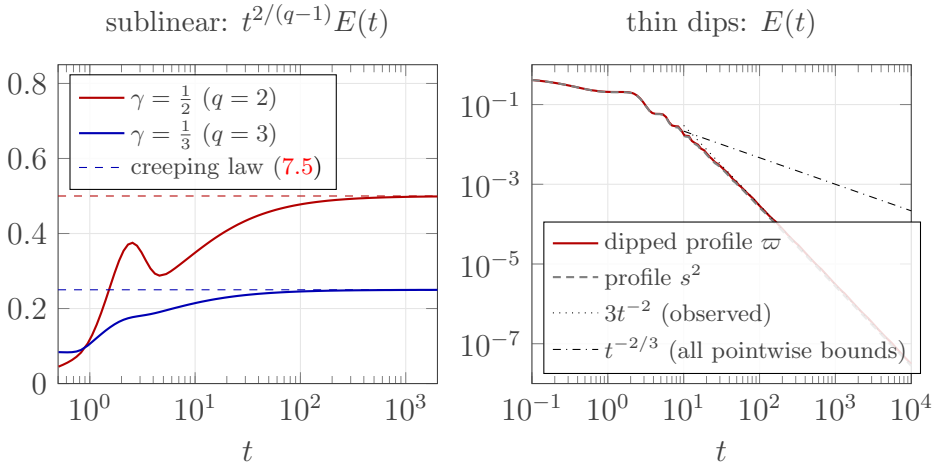



FIGURE 13. Left (Example 7.3): for the non-Lipschitz feedback $|v|^{\gamma-1}v$, the compensated energy converges to the constant of the creeping law (7.5), so the exponent $2\gamma/(1-\gamma)$ given by Theorem 4.6 through $g_2 = s^{1/\gamma}$ is sharp. Right (Example 7.4): thin dips of the feedback lower the envelope to $\eta^3$ and every pointwise bound to $t^{-2/3}$, while the solution decays like $t^{-2}$.

7.5. **What is sharp and what is not.** Table 1 summarizes the examples. The exponents given by Theorem 4.6 are attained in all monotone examples, uniformly in the discretization parameter, including oscillating and non-Lipschitz profiles. The multiplicative constants are far from optimal. They arise from three sources: the factor $K = \max\{2, 4c_A\}$ of the comparison, the factor $\rho = \delta/(2K)$, which converts one step of length $T$ into a time unit, and the choice $\beta = 8c_B mT/\delta$, which makes the error term of Lemma 3.1 absorbable for *all* data at once. This is the price of bounds that do not depend on the data, on $\|C\|$ or on the mesh. Example 7.4 indicates, numerically, that the exponent itself can be lost when the feedback has thin dips, and suggests that the correct modulus is an averaged one (Section 8).

The computed figures were produced with a short Python script (NumPy, SciPy and Matplotlib), supplied as supplementary material with this article; it reproduces Figures 4, 5, 6, 8, 12 and 13, the constants of Section 7 and the values of Table 1.

TABLE 1. Explicit bounds versus computed decay ($t \to \infty$). For the bounds we list the leading term of (7.3) or (7.4); "observed" is the range of the compensated energy on the time window of the computation.

| example | bound of Theorem 4.6 | observed | exponent |
|---|---|---|---|
| string, $p = 3$ (Ex. 7.2) | $1024\pi(\ln 6)^2\, t^{-1} \approx 1.0 \cdot 10^4\, t^{-1}$ | $(1.36\text{–}1.53)\, t^{-1}$ | sharp |
| string, $p = 2$ (Ex. 7.2) | $131072\pi(\ln 6)^3\, t^{-2} \approx 2.4 \cdot 10^6\, t^{-2}$ | $(5.3\text{–}6.2)\, t^{-2}$ | sharp |
| oscillator, $\gamma = \frac{1}{2}$ (Ex. 7.3) | $131072(\ln 6)^3\, t^{-2} \approx 7.5 \cdot 10^5\, t^{-2}$ | $\to 0.5\, t^{-2}$ | sharp |
| oscillator, $\gamma = \frac{1}{3}$ (Ex. 7.3) | $1024(\ln 6)^2\, t^{-1} \approx 3.3 \cdot 10^3\, t^{-1}$ | $\to 0.25\, t^{-1}$ | sharp |
| oscillating $g$ (Ex. 6.2) | $C\, t^{-2}$ | $(2.8\text{–}5.9)\, t^{-2}$ | sharp |
| midpoint, cubic (Ex. 5.6) | $C\, (k\Delta t)^{-1}$ | $\le 1.05\, (k\Delta t)^{-1}$ | sharp |
| thin dips (Ex. 7.4) | $C\, t^{-2/3}$ | $(2.4\text{–}3.2)\, t^{-2}$ | *not* sharp |

## 8. Discussion and open problems

Table 2 compares the hypotheses and conclusions of the present paper with those of [7], and Figure 14 summarizes the situations covered by each approach.

| | [7] | present paper |
|---|---|---|
| linear hypothesis | observability of the conservative system | exponential decay of the linear damped system (implied by the former, Props. 4.14, 5.7) |
| constants depend on | $T^2\|B\|^2$, $C_T$, $g$, $E(0)$ | $\delta, T, m, g_1, g_2$; data only through a log shift; not on $\|C\|$ |
| feedback profile | $g \in C^1$, $g'(0) = 0$, $H$ strictly convex, $sg'(s)^2/g(s) \to 0$ | two continuous nondecreasing profiles $g_1, g_2$; monotone envelope $h$ |
| feedback regularity | locally Lipschitz | monotone, or locally Lipschitz, or $\dim X < \infty$ |
| nonlinear estimate | Jensen on sublevel sets | pointwise level-set inequality (Lemma 3.1) |
| space discretization | (34)–(35) (fail for $P_1$) | nodal structure (6.1) |
| full discretization | sketched | Corollary 6.6 |
| boundary feedback (discrete) | open | Corollary 6.8 |

TABLE 2. Comparison with [7].

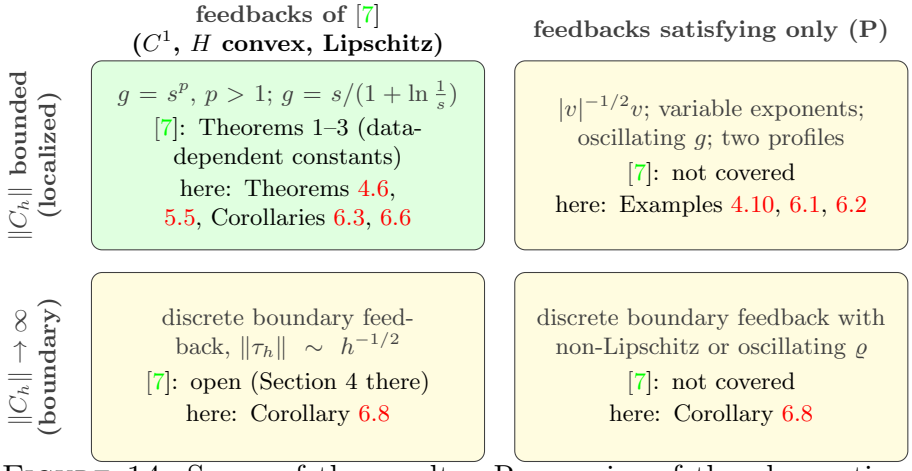



FIGURE 14. Scope of the results. Rows: size of the observation operator after discretization; columns: class of feedbacks. Only the upper-left cell is covered by [7]; all four are covered here, with constants independent of $\|C_h\|$ and of the data (up to a logarithmic waiting time).

We conclude with four directions that the present method leaves open. The first concerns full discretization with boundary feedback. Corollary 6.8 is a space semi-discrete result. Corollary 6.6 relies on [14, Theorem 1.1], whose constants depend on a uniform bound for the damping operator, and this fails when $\|\tau_h\| \to \infty$. What is missing is a uniform-in-$h$ version of the time-discretization theorem for linear systems with unboundedly growing feedback. In the continuous setting a result of this type for a fixed unbounded $B$ was obtained in [10] under a resolvent hypothesis on $B^*(\lambda - A)^{-1}B$; a version uniform in $h$ would immediately give the fully discrete nonlinear result through Theorem 5.5.

A second question is the case of an unbounded $C$ in continuous time. The proof of Lemma 4.5 is formally valid for admissible unbounded $C$, such as boundary damping of the wave equation. A rigorous version requires that the identity for $\frac{\mathrm{d}}{\mathrm{d}t}\|u - z\|^2$ hold along strong solutions. This is the case for standard boundary-damped wave equations, but we have not carried out the verification in an abstract framework; the continuous boundary case is in any event covered by the direct method [3, 4].

A third limitation is the need for exponential linear stability. The argument requires a uniform gain $\delta$ for all initial data. If the linear damped system is only polynomially stable (no geometric control condition, indirect damping), the inequality $\|z(T)\|^2 \le (1 - \delta)\|z(0)\|^2$ holds only after a loss of regularity, and the comparison loses its strength. Paunonen and Seifert [23] obtain polynomial rates for $x' = Ax - B\varphi(B^*x)$ under non-uniform observability by different means, and polynomial decay estimates for feedback-stabilized bilinear neutral systems with distributed delay in Banach spaces are derived in [15]; combining their approach with discretization, uniformly in the mesh, is open.

Finally, lower bounds remain incomplete. Proposition 6.9 shows that the polynomial exponent cannot be improved in general, and [4] proves optimality in finite dimension for $\limsup \Lambda_H < 1$. Matching lower bounds for strong solutions, under an $L^\infty$-interpolation property of the velocity, are proved in [1] for near-linear profiles. The envelope rate $\Phi^{-1}$ is sharp in all the monotone examples of Section 7, but not for feedbacks with thin dips (Example 7.4). Finding the right averaged modulus, and proving the corresponding upper and lower bounds, is open.

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