Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 9 (2022) 52-66

TOPOLOGICAL INDICES OF FOUR NEW TENSOR PRODUCTS OF GRAPHS

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Abstract. In this paper, we obtain multiplicative Zagreb indices and first neighbourhood Zagreb index of $\mathcal{F}$-tensor products (four new tensor products based on transformations of a graph) of graphs.

1. Introduction

The topological indices are graph invariants which are numerical values associated with molecular graphs. In mathematical chemistry, molecular descriptors play a leading role specifically in the field of QSPR/QSAR modelling.

There are several papers devoted to topological indices of graph operations. Among them Muruganandam et al. obtained the multiplicative Zagreb indices of products of graphs in [21]. Khalifeh et al. [15] obtained first and second Zagreb indices of graph operations. Basavanagoud et al. [5] obtained hyper Zagreb index of graph operations. Nadjafi-Arani et al. obtained degree distance based topological indices of tensor products of graphs in [19]. For some other topological indices of graph operations one can refer to [7, 22, 23, 24].

2. Definitions and Preliminaries

Let G be a finite undirected graph without loops and multiple edges on n vertices and m edges and is called (n, m) graph. We denote vertex set and edge set of graph G as $V(G)$ and $E(G)$, respectively. For a graph G, the degree of a vertex v is the number of edges incident to v and is denoted by $d_G(v)$. The neighbourhood of a vertex $u \in V(G)$ is defined as the set $N_G(u)$ consisting of all vertices v which are adjacent with u. The degree of a vertex $u \in V(G)$, denoted by $d_G(u)$ and is equal to $|N_G(u)|$. Let $S_G(v) = \sum\limits_{u \in N_G(u)} d_G(u)$ be the degree sum of neighbourhood vertices where, $N_G(v) = \{u; uv \in E(G)\}$.

Date: Received: Aug 22, 2022; Accepted: Oct 15, 2022.

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2010 *Mathematics Subject Classification.* Primary 05C07; Secondary 05C76, 92E10.

Key words and phrases. Tensor products, multiplicative Zagreb indices, first neighbourhood Zagreb index.

For a molecular graph G, The first and second Zagreb index are in [12] and defined as

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2,$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The first Zagreb index [20] can also be expressed as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)].$$

Another degree-based graph invariant called forgotten topological index or F-index was introduced by Furtula and Gutman [9] is defined as

$$F(G) = \sum_{v\in V(G)} d_G(u)^3 = \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2].$$

Li and Zheng [17] introduced the concept of first general Zagreb index $M_1^\alpha(G)$ of G as

$$M_1^\alpha(G) = \sum_{v\in V(G)} d_G(v)^\alpha.$$

The hyper-Zagreb index was introduced by Shirdel et al. in [26] which is defined as

$$HM(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^2.$$

The sum-connectivity index of a graph G was defined in [31] as

$$\chi(G) = \sum_{xy\in E(G)} [d_G(x) + d_G(y)]^{-\frac{1}{2}}.$$

Further, it has been extended to the general sum-connectivity index which is defined in [32] as

$$\chi_\alpha(G) = \sum_{xy\in E(G)} [d_G(x) + d_G(y)]^\alpha.$$

For $\alpha = 3$, we have

$$\chi_3(G) = \sum_{xy\in E(G)} [d_G(x) + d_G(y)]^3.$$

The fifth M-Zagreb indices [16] of a molecular graph G are defined as

$$M_1G_5(G) = \sum_{uv\in E(G)} (S_G(u) + S_G(v)),$$

$$M_2G_5(G) = \sum_{uv\in E(G)} (S_G(u)S_G(v)).$$

The fifth hyper M-Zagreb indices [16] of a molecular graph G are defined as

$$HM_1G_5(G) = \sum_{uv\in E(G)} (S_G(u) + S_G(v))^2,$$

$$HM_2G_5(G) = \sum_{uv \in E(G)} (S_G(u)S_G(v))^2.$$

In [2] Basavanagoud et al. defined first neighbourhood Zagreb index (FNZI) of a graph as

$$NM_1(G) = \sum_{v \in V(G)} S_G(v)^2.$$

In 1984, Narumi and Katayama [18] considered the product index as

$$NK(G) = \prod_{u \in V(G)} d_G(u).$$

Tomović and Gutman renamed this molecular structure descriptor as the Narumi-Katayama index [29]. In 2010, Todeshine et al. [27, 28] proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows

$$\prod\nolimits_1(G) = \prod_{u \in V(G)} d_G(u)^2 = [NK(G)]^2,$$

$$\prod\nolimits_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v).$$

These two graph invariants are called first and second multiplicative Zagreb indices by Gutman [11]. Recently, Eliasi et al. [8] introduced a further multiplicative version of the first Zagreb index as

$$\prod\nolimits_1^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)].$$

In [10, 30] the authors called it as multiplicative sum Zagreb index and modified first multiplicative Zagreb index respectively. The second multiplicative Zagreb index for any graph G can also be written as [11].

$$\prod\nolimits_2(G) = \prod_{u \in V(G)} d_G(u)^{d_G(u)}.$$

Basavanagoud et al.[4] introduced a further multiplicative version of the second Zagreb index as

$$\prod\nolimits_2^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)]^{d_G(u)+d_G(v)}.$$

The main properties of multiplicative Zagreb indices are summarized in [7, 30].

For a graph G with vertex set $V(G)$ and edge set $E(G)$. Let $L(G)$ be the line graph of G. There are four related graphs as follows (see fig. 1).

(i) The subdivision graph $S = S(G)$ [13]; is the graph obtained by inserting a new vertex onto each edge of G;
(ii) Semitotal-point graph $T_2 = T_2(G)$ [25]; $V(T_2) = V(G) \cap E(G)$ and $E(G) = E(S) \cap E(G)$;
(iii) Semitotal-line graph $T_1 = T_1(G)$ [25]; $V(T_1) = V(G) \cap E(G)$ and $E(T_1) = E(S) \cap E(L)$;
(iv) Total graph $T = T(G)$ [6]; $V(T) = V(G) \cap E(G)$ and $E(T) = E(S) \cap E(G) \cap E(L)$. Here $L = L(G)$ is the line graph of G.

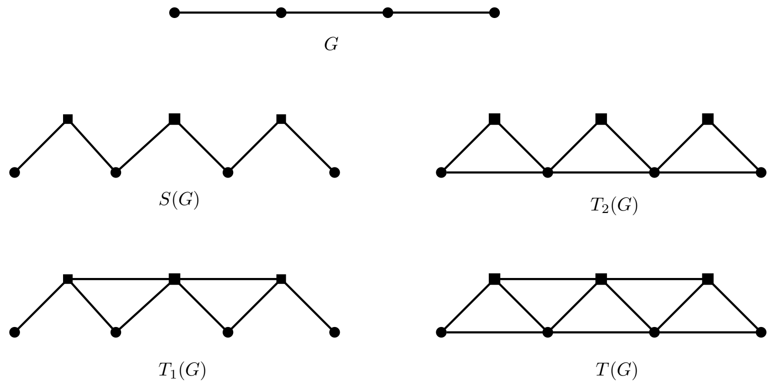

FIGURE 1. Graph G and its transformations $S(G)$, $T_2(G)$, $T_1(G)$, $T(G)$.

The tensor products [14] $G_1 \otimes G_2$ of two graphs G_1 and G_2 of order n_1 and n_2, respectively, is defined as the graph with vertex set $V_1 \times V_2$ and (u_1, v_1) is adjacent with (u_2, v_2) if and only if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

Recently, Basavanagoud et al. introduced four new tensor products of graphs by extending tensor product as follows:

Definition 2.1. [3] Let $\mathcal{F}$ be one of the symbols S, T_2, T_1 or T. The $\mathcal{F}$- tensor product $G_1 \otimes G_2$ is a graph with the set of vertices $V(G_1 \otimes G_2) = ((V(G_1) \cup E(G_1)) \times V(G_2))$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \otimes_{\mathcal{F}} G_2$ are adjacent if and only if u_1 is adjacent to v_1 in $E(\mathcal{F}(G_1))$ and u_2 is adjacent to v_2 in G_2.

Thus authors in [3] obtained four new graph operations namely $G_1 \otimes_S G_2$, $G_1 \otimes_{T_2} G_2$, $G_1 \otimes_{T_1} G_2$ and $G_1 \otimes_T G_2$ as depicted in Fig. 2 and studied the first and second Zagreb indices of these graphs.

In the recent paper [1] Basavanagoud et al. obtained F-index and hyper-Zagreb index of four new tensor products of graphs. This motivates us to study topological indices of $\mathcal{F}$-tensor products of graphs in this paper.

Before, we proceed to the purpose of the paper, we mention some existing results.

Theorem 2.2. [4] Let G be a graph of order n and size m, then

(i) $\prod_1(S) = 4^m \prod_1(G)$,
(ii) $\prod_1(T_2) = 4^{n+m} \prod_1(G)$,
(iii) $\prod_1(T_1) = \prod_1(G)[\prod_1^*(G)]^2$,
(iv) $\prod_1(T) = 4^n \prod_1(G)[\prod_1^*(G)]^2$.

Theorem 2.3. [4] Let G be a graph of order n and size m, then

(i) $\prod_2(S) = 4^m \prod_2(G)$,
(ii) $\prod_2(T_2) = 64^m \prod_1(G) \prod_2(G)$,

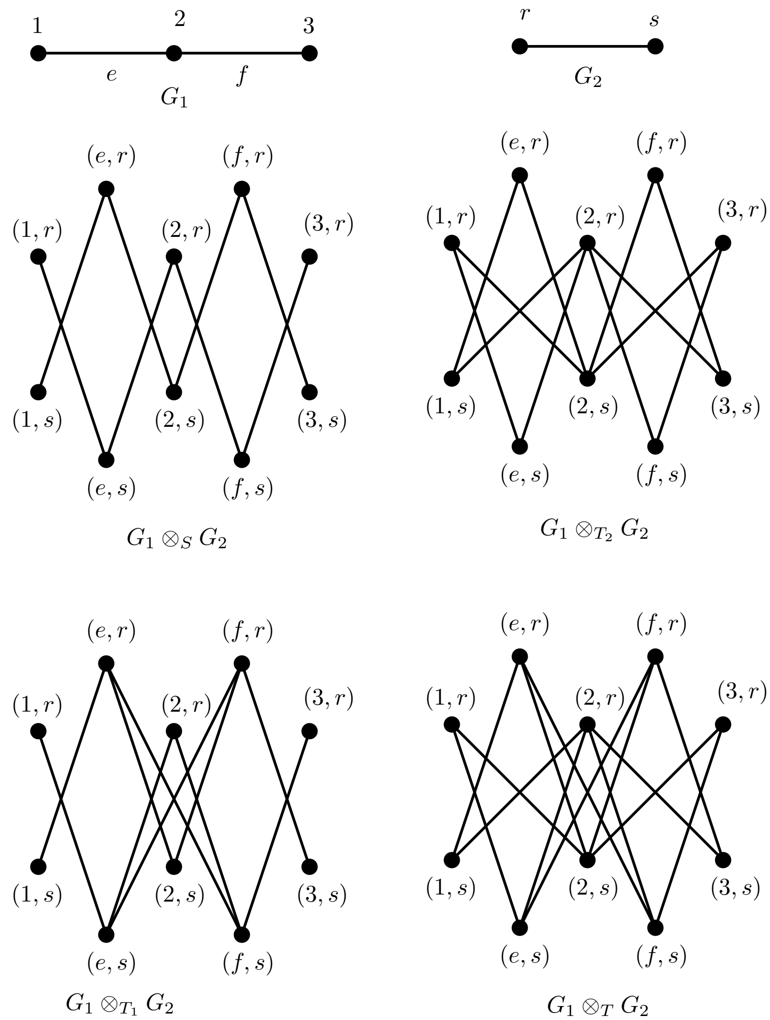

FIGURE 2. Graphs G_1, G_2 and $G_1 \otimes_{\mathcal{F}} G_2$

(iii) $\prod_2(T_1) = \prod_2(G)\prod_2^*(G)$,
(iv) $\prod_2(T) = 16^m \prod_2^*(G)[\prod_2(G)]^2$.

3. Main Results

Let $G_i = (V_i, E_i)$ with $|V_i| = n_i$ and $|E_i| = m_i$ be two graphs for i=1, 2. In this section, we obtain first, second multiplicative Zagreb indices and first neighbourhood Zagreb index of $\mathcal{F}$-tensor products of graphs.

3.1. **First multiplicative Zagreb index of $\mathcal{F}$-tensor products of graphs.**

In this section, we proceed to obtain first multiplicative Zagreb index of $\mathcal{F}$-tensor products of graphs, where $\mathcal{F} \in \{S, T_2, T_1, T\}$.

The following theorem gives the first multiplicative Zagreb index of S-tensor products of two graphs G_1 and G_2.

Theorem 3.1. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod\nolimits_1(G_1 \otimes_S G_2) = 4^{m_1 n_2}\ [\prod\nolimits_1 (G_1)]^{n_2}[\prod\nolimits_1(G_2)]^{2+n_1}.$$

Proof. By definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod\nolimits_1(G_1 \otimes_S G_2) &= \prod_{(u,v)\in V(G_1\otimes_S G_2)} d^2_{G_1\otimes_S G_2}(u,v)\\ &= \prod_{u\in(S(G_1))\cap V(G_1)}\ \prod_{v\in V(G_2)} [d_{S(G_1)}(u)d_{G_2}(v)]^2\times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{e\in(S(G_1))\cap E(G_1)} [d_{S(G_1)}(e)d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (S(G_1)) \cap V(G_1)$, $d_{S(G_1)}(u) = d_{G_1}(u)$ and $e \in (S(G_1)) \cap E(G_1)$, $d_{S(G_1)}(e) = 2$. Therefore,

$$\begin{aligned}\prod\nolimits_1(G_1 \otimes_S G_2) &= \prod_{u\in V(G_1)}\ \prod_{v\in V(G_2)} [d_{G_1}(u)d_{G_2}(v)]^2\times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{e\in E(G_1)} [2d_{G_2}(v)]^2\\ &= \prod_{u\in V(G_1)}\ \prod_{v\in V(G_2)} [d^2_{G_1}(u)d^2_{G_2}(v)] \prod_{v\in V(G_2)}\ \prod_{e\in E(G_1)} [4d^2_{G_2}(v)]\\ &= [\prod\nolimits_1(G_1)]^{n_2}[\prod\nolimits_1(G_2)]^{n_1} 4^{m_1 n_2}[\prod\nolimits_1(G_2)]^2\\ &= 4^{m_1 n_2}[\prod\nolimits_1(G_1)]^{n_2}[\prod\nolimits_1(G_2)]^{2+n_1}.\end{aligned}$$ ∎

The following theorem gives the first multiplicative Zagreb index of T_2-tensor products of two graphs G_1 and G_2.

Theorem 3.2. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod\nolimits_1 (G_1 \otimes_{T_2} G_2) = 4^{n_1 n_2}\ 4^{m_1 n_2}\ [\prod\nolimits_1 (G_1)]^{n_2}[\prod\nolimits_1(G_2)]^{2+n_1}.$$

Proof. By definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod\nolimits_1(G_1 \otimes_{T_2} G_2) &= \prod_{(u,v)\in V(G_1\otimes_{T_2} G_2)} d^2_{G_1\otimes_{T_2} G_2}(u,v)\\ &= \prod_{u\in(T_2(G_1))\cap V(G_1)}\ \prod_{v\in V(G_2)} [d_{T_2(G_1)}(u)d_{G_2}(v)]^2\times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{e\in(T_2(G_1))\cap E(G_1)} [d_{T_2(G_1)}(e)d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (T_2(G_1)) \cap V(G_1)$, $d_{T_2(G_1)}(u) = 2d_{G_1}(u)$ and $e \in (T_2(G_1)) \cap E(G_1)$, $d_{T_2(G_1)}(e) = 2$. Therefore,

$$\begin{aligned}\prod\nolimits_1(G_1 \otimes_{T_2} G_2) &= \prod_{u\in V(G_1)}\ \prod_{v\in V(G_2)} [2d_{G_1}(u)d_{G_2}(v)]^2\times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{e\in E(G_1)} [2d_{G_2}(v)]^2\\ &= \prod_{u\in V(G_1)}\ \prod_{v\in V(G_2)} [4d^2_{G_1}(u)d^2_{G_2}(v)] \prod_{v\in V(G_2)}\ \prod_{e\in E(G_1)} [4d^2_{G_2}(v)]\end{aligned}$$

$$= 4^{n_1n_2}[\prod_1(G_1)]^{n_2}[\prod_1(G_2)]^{n_1}4^{m_1n_2}[\prod_1(G_2)]^2$$

$$= 4^{n_1n_2}\ 4^{m_1n_2}[\prod_1(G_1)]^{n_2}[\prod_1(G_2)]^{2+n_1}.$$

□

The following theorem gives the first multiplicative Zagreb index of T_1-tensor products of two graphs G_1 and G_2.

Theorem 3.3. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then* $\prod_1(G_1 \otimes_{T_1} G_2) = [\prod_1(G_1)]^{n_2}[\prod_1(G_2)]^{2+n_1}[\prod_1^*(G_1)]^{2n_2}$.

Proof. By definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod_1(G_1 \otimes_{T_1} G_2) &= \prod_{(u,v)\in V(G_1\otimes_{T_1}G_2)} d^2_{G_1 \bigotimes_{T_1} G_2}(u,v)\\ &= \prod_{u\in(T_1(G_1))\cap V(G_1)}\ \prod_{v\in V(G_2)} [d_{T_1(G_1)}(u)d_{G_2}(v)]^2 \times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{e\in(T_1(G_1))\cap E(G_1)} [d_{T_1(G_1)}(e)d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (T_1(G_1))\cap V(G_1), d_{T_1(G_1)}(u) = d_{G_1}(u)$ and $e \in (T_1(G_1))\cap E(G_1), d_{T_1(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$ Therefore,

$$\begin{aligned}\prod_1(G_1 \otimes_{T_1} G_2) &= \prod_{u\in V(G_1)}\ \prod_{v\in V(G_2)} [d_{G_1}(u)d_{G_2}(v)]^2 \times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{xy\in E(G_1)} [(d_{G_1}(x) + d_{G_1}(y))d_{G_2}(v)]^2\\ &= \prod_{u\in V(G_1)}\ \prod_{v\in V(G_2)} [d^2_{G_1}(u)d^2_{G_2}(v)] \prod_{v\in V(G_2)}\ \prod_{xy\in E(G_1)} [d_{G_1}(x) +\end{aligned}$$

$d_{G_1}(y)]^2 d^2_{G_2}(v)$

$$\begin{aligned}&= [\prod_1(G_1)]^{n_2}[\prod_1(G_2)]^{n_1}[\prod_1^*(G_1)]^{2n_2}[\prod_1(G_2)]^2\\ &= [\prod_1(G_1)]^{n_2}[\prod_1(G_2)]^{2+n_1}[\prod_1^*(G_1)]^{2n_2}.\end{aligned}$$

□

The following theorem gives the first multiplicative Zagreb index of T-tensor products of two graphs G_1 and G_2.

Theorem 3.4. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then* $\prod_1(G_1 \otimes_T G_2) = 4^{n_1n_2}\ [\prod_1(G_1)]^{n_2}[\prod_1(G_2)]^{2+n_1}\ [\prod_1^*(G_1)]^{2n_2}$.

Proof. By definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod_1(G_1 \otimes_T G_2) &= \prod_{(u,v)\in V(G_1 \bigotimes_T G_2)} d^2_{G_1\otimes_T G_2}(u,v)\\ &= \prod_{u\in(T(G_1))\cap V(G_1)}\ \prod_{v\in V(G_2)} [d_{T(G_1)}(u)d_{G_2}(v)]^2 \times\\ &\quad \prod_{v\in V(G_2)}\ \prod_{e\in(T(G_1))\cap E(G_1)} [d_{T(G_1)}(e)d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (T(G_1)) \cap V(G_1)$, $d_{T(G_1)}(u) = 2d_{G_1}(u)$ and $e \in (T(G_1)) \cap E(G_1)$, $d_{T(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$. Therefore,

$$\begin{aligned}\prod\nolimits_1(G_1 \otimes_T G_2) &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u)d_{G_2}(v)]^2 \times \\ &\quad \prod_{v\in V(G_2)} \prod_{xy\in E(G_1)} [(d_{G_1}(x) + d_{G_1}(y))d_{G_2}(v)]^2 \\ &= 4^{n_1n_2}[\prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d^2_{G_1}(u)d^2_{G_2}(v)]][\prod\nolimits_1^*(G_1)]^{2n_2}[\prod\nolimits_1(G_2)]^2 \\ &= 4^{n_1n_2}[\prod\nolimits_1(G_1)]^{n_2}[\prod\nolimits_1(G_2)]^{n_1}[\prod\nolimits_1^*(G_1)]^{2n_2}[\prod\nolimits_1(G_2)]^2 \\ &= 4^{n_1n_2}[\prod\nolimits_1(G_1)]^{n_2}[\prod\nolimits_1(G_2)]^{2+n_1}[\prod\nolimits_1^*(G_1)]^{2n_2}.\end{aligned}$$

□

3.2. **Second multiplicative Zagreb index of $\mathcal{F}$-tensor products of graphs.** In this section, we proceed to obtain second multiplicative Zagreb index of $\mathcal{F}$-tensor products of graphs, where $\mathcal{F} \in \{S, T_2, T_1, T\}$.

The following theorem gives the second multiplicative Zagreb index of S-tensor products of two graphs G_1 and G_2.

Theorem 3.5. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod\nolimits_2 (G_1 \otimes_S G_2) = 16^{m_1m_2} [\prod\nolimits_2 (G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{4m_1}.$$

Proof. By definition of second multiplicative Zagreb index, we have

$$\begin{aligned}\prod\nolimits_2(G_1 \otimes_S G_2) &= \prod_{(u,v)\in V(G_1\otimes_S G_2)} [d_{G_1\otimes_S G_2}(u, v)]^{d_{G_1\otimes_S G_2}(u,v)} \\ &= \prod_{u\in(S(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{S(G_1)}(u)d_{G_2}(v)]^{d_{S(G_1)}(u)d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{e\in(S(G_1))\cap E(G_1)} [d_{S(G_1)}(e)d_{G_2}(v)]^{d_{S(G_1)}(e)d_{G_2}(v)}.\end{aligned}$$

For $u \in (S(G_1)) \cap V(G_1)$, $d_{S(G_1)}(u) = d_{G_1}(u)$ and $e \in (S(G_1)) \cap E(G_1)$, $d_{S(G_1)}(e) = 2$. Therefore,

$$\begin{aligned}\prod\nolimits_2(G_1 \otimes_S G_2) &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u)d_{G_2}(v)]^{d_{G_1}(u)d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{e\in E(G_1)} [2d_{G_2}(v)]^{2d_{G_2}(v)} \\ &= [\prod\nolimits_2(G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{2m_1} 4^{m_1m_2}[\prod\nolimits_2(G_2)]^{m_1} \\ &= 16^{m_1m_2}[\prod\nolimits_2(G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{4m_1}.\end{aligned}$$

□

The following theorem gives the second multiplicative Zagreb index of T_2-tensor products of two graphs G_1 and G_2.

Theorem 3.6. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod\nolimits_2 (G_1 \otimes_{T_2} G_2) = [4096]^{m_1m_2} [\prod\nolimits_2 (G_1)]^{4m_2}[\prod\nolimits_2(G_2)]^{6m_1}.$$

Proof. By definition of second multiplicative Zagreb index, we have

$$\begin{aligned}\prod\nolimits_2(G_1 \otimes_{T_2} G_2) &= \prod_{(u,v)\in V(G_1\otimes_{T_2}G_2)} [d_{G_1\otimes_{T_2}G_2}(u,v)]^{d_{G_1}\otimes_{T_2}G_2(u,v)}\\ &= \prod_{u\in(T_2(G_1))\cap V(G_1)}\prod_{v\in V(G_2)} [d_{T_2(G_1)}(u)d_{G_2}(v)]^{d_{T_2(G_1)}(u)d_{G_2}(v)}\times\\ &\quad \prod_{v\in V(G_2)}\prod_{e\in(S(G_1))\cap E(G_1)} [d_{T_2(G_1)}(e)d_{G_2}(v)]^{d_{T_2(G_1)}(e)d_{G_2}(v)}.\end{aligned}$$

For $u \in (T_2(G_1)) \cap V(G_1)$, $d_{T_2(G_1)}(u) = 2d_{G_1}(u)$ and $e \in (T_2(G_1)) \cap E(G_1)$, $d_{T_2(G_1)}(e) = 2$. Therefore,

$$\begin{aligned}\prod\nolimits_2(G_1 \otimes_{T_2} G_2) &= \prod_{u\in V(G_1)}\prod_{v\in V(G_2)} [2d_{G_1}(u)d_{G_2}(v)]^{2d_{G_1}(u)d_{G_2}(v)}\times\\ &\quad \prod_{v\in V(G_2)}\prod_{e\in E(G_1)} [2d_{G_2}(v)]^{2d_{G_2}(v)}\\ &= 4^{m_1m_2}[\prod\nolimits_2(G_1)]^{4m_2}[\prod\nolimits_2(G_2)]^{4m_1}4^{2m_1m_2}[\prod\nolimits_2(G_2)]^{2m_1}\\ &= [4096]^{m_1m_2}[\prod\nolimits_2(G_1)]^{4m_2}[\prod\nolimits_2(G_2)]^{6m_1}. \qquad \square\end{aligned}$$

The following theorem gives the second multiplicative Zagreb index of T_1-tensor products of two graphs G_1 and G_2.

Theorem 3.7. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*
$$\prod\nolimits_2 (G_1 \otimes_{T_1} G_2) = [\prod\nolimits_2 (G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{M_1(G_1)+2m_1}\ [\prod\nolimits_2^* (G_1)]^{2m_2}.$$

Proof. By definition of second multiplicative Zagreb index, we have

$$\begin{aligned}\prod\nolimits_2(G_1 \otimes_{T_1} G_2) &= \prod_{(u,v)\in V(G_1\otimes_{T_1}G_2)} [d_{G_1\otimes_{T_1}G_2}(u,v)]^{d_{G_1}\otimes_{T_1}G_2(u,v)}\\ &= \prod_{u\in(T_1(G_1))\cap V(G_1)}\prod_{v\in V(G_2)} [d_{T_1(G_1)}(u)d_{G_2}(v)]^{d_{T_1(G_1)}(u)d_{G_2}(v)}\times\\ &\quad \prod_{v\in V(G_2)}\prod_{e\in(T_1(G_1))\cap E(G_1)} [d_{T_1(G_1)}(e)d_{G_2}(v)]^{d_{T_1(G_1)}(e)d_{G_2}(v)}.\end{aligned}$$

For $u \in (T_1(G_1)) \cap V(G_1)$, $d_{T_1(G_1)}(u) = d_{G_1}(u)$ and $e \in (T_1(G_1)) \cap E(G_1)$, $d_{T_1(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$. Therefore,

$$\begin{aligned}\prod\nolimits_2(G_1 \otimes_{T_1} G_2) &= \prod_{u\in V(G_1)}\prod_{v\in V(G_2)} [d_{G_1}(u)d_{G_2}(v)]^{d_{G_1}(u)d_{G_2}(v)}\times\\ &\quad \prod_{v\in V(G_2)}\prod_{xy\in E(G_1)} [(d_{G_1}(x){+}d_{G_1}(y))d_{G_2}(v)]^{(d_{G_1}(x)+d_{G_1}(y))d_{G_2}(v)}\\ &= [\prod\nolimits_2(G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{2m_1}[\prod\nolimits_2^*(G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{M_1(G_1)}\\ &= [\prod\nolimits_2(G_1)]^{2m_2}[\prod\nolimits_2(G_2)]^{M_1(G_1)+2m_1}[\prod\nolimits_2^*(G_1)]^{2m_2}. \qquad \square\end{aligned}$$

The following theorem gives the second multiplicative Zagreb index of T-tensor products of two graphs G_1 and G_2.

Theorem 3.8. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_2 (G_1 \otimes_T G_2) = 256^{m_1 m_2} \, [\prod_2 (G_1)]^{4m_2} [\prod_2 (G_2)]^{4m_1 + M_1(G_1)} \, [\prod_2^* (G_1)]^{2m_2}.$$

Proof. By definition of second multiplicative Zagreb index, we have

$$\begin{aligned}\prod_2(G_1 \otimes_T G_2) &= \prod_{(u,v)\in V(G_1 \bigotimes_T G_2)} [d_{G_1\otimes_T G_2}(u,v)]^{d_{G_1\otimes_T G_2}(u,v)} \\ &= \prod_{u\in(T(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{T(G_1)}(u) d_{G_2}(v)]^{d_{T(G_1)}(u) d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{e\in(S(G_1))\cap E(G_1)} [d_{T(G_1)}(e) d_{G_2}(v)]^{d_{T(G_1)}(e) d_{G_2}(v)}.\end{aligned}$$

For $u \in (T(G_1)) \cap V(G_1), d_{T(G_1)}(u) = 2d_{G_1}(u)$ and for $e \in (T(G_1)) \cap E(G_1)$, $d_{S(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$ Therefore,

$$\begin{aligned}\prod_2(G_1 \otimes_T G_2) &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) d_{G_2}(v)]^{2d_{(G_1)}(u) d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{xy\in E(G_1)} [(d_{G_1}(x)+d_{G_1}(y)) d_{G_2}(v)]^{(d_{G_1}(x)+d_{G_1}(y)) d_{G_2}(v)} \\ &= 4^{4m_1 m_2} [\prod_2(G_1)]^{4m_2} [\prod_2(G_2)]^{4m_1} [\prod_2^*(G_1)]^{2m_2} [\prod_2(G_2)]^{M_1(G_1)} \\ &= 256^{m_1 m_2} [\prod_2(G_1)]^{4m_2} [\prod_2(G_2)]^{4m_1+M_1(G_1)} [\prod_2^*(G_1)]^{2m_2}.\end{aligned}$$

□

3.3. **First neighbourhood Zagreb index of $\mathcal{F}$-tensor products of graphs.**

In this section, we proceed to obtain first neighbourhood Zagreb index of $\mathcal{F}$-tensor products of graphs, where $\mathcal{F} \in \{S, T_2, T_1, T\}$.

The following theorem gives the first neighbourhood Zagreb index of S-tensor products of two graphs G_1 and G_2.

Theorem 3.9. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$NM_1(G_1 \otimes_S G_2) = 4M_1(G_1) NM_1(G_2) + HM(G_1) NM_1(G_2).$$

Proof. By definition of first neighbourhood Zagreb index, we have

$$NM_1(G_1 \otimes_S G_2) = \sum_{(u,v)\in V(G_1\otimes_S G_2)} S_{G_1\otimes_S G_2}(u,v)^2$$

$$= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_S G_2}(u,v)^2 + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_S G_2}(e,v)^2$$

$$= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big(2d_{G_1}(u) S_{G_2}(v)\Big)^2 + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} \Big(S_{G_2}(v)(d_{G_1}(u) + d_{G_1}(v))\Big)^2.$$

Expanding and by applying summation, we get

$$NM_1(G_1 \otimes_S G_2) = 4M_1(G_1) NM_1(G_2) + HM(G_1) NM_1(G_2).$$

□

The following theorem gives the first neighbourhood Zagreb index of T_2-tensor products of two graphs G_1 and G_2.

Theorem 3.10. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} NM_1(G_1 \otimes_{T_2} G_2) &= 4M_1(G_1)NM_1(G_2) + 4NM_1(G_1)NM_1(G_2) + 4NM_1(G_2)HM(G_1) \\ &\quad +16M_2(G_1)NM_1(G_2). \end{aligned}$$

Proof. By definition of first neighbourhood Zagreb index, we have

$$NM_1(G_1 \otimes_{T_2} G_2) = \sum_{(u,v)\in V(G_1\otimes_{T_2}G_2)} S_{G_1\otimes_{T_2}G_2}(u,v)^2$$

$$\begin{aligned} &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_{T_2}G_2}(u,v)^2 + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_{T_2}G_2}(e,v)^2 \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big(2d_{G_1}(u)S_{G_2}(v) + 2S_{G_1}(u)S_{G_2}(v)\Big)^2 \\ &\quad + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} \Big(2S_{G_2}(v)(d_{G_1}(u) + d_{G_2}(v))\Big)^2. \end{aligned}$$

Expanding and by applying summation, we get

$$\begin{aligned} NM_1(G_1 \otimes_{T_2} G_2) &= 4M_1(G_1)NM_1(G_2) + 4NM_1(G_1)NM_1(G_2) + 4NM_1(G_2)HM(G_1) \\ &\quad +16M_2(G_1)NM_1(G_2). \end{aligned}$$

□

The following theorem gives the first neighbourhood Zagreb index of T_1-tensor products of two graphs G_1 and G_2.

Theorem 3.11. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} NM_1(G_1 \otimes_{T_1} G_2) &= NM_1(G_2)M_1(G_1) + NM_1(G_1)NM_1(G_2) + NM_1(G_2)HM(G_1) \\ &\quad +M_1^4(G_2)\chi_4(G_1) + 2NM_1(G_2)\sum_{u\in V(G_1)} d_{G_1}^2(u)S_{G_1}(u) \\ &\quad +M_1^4(G_2)\sum_{uv\in E(G_1)} (S_{G_1}(u) + S_{G_1}(v))^2(d_{G_1}(u) + d_{G_1}(v))^2 \\ &\quad +2\sum_{v\in V(G_2)} S_2(v)d_{G_2}(v)\sum_{uv\in E(G_1)} (S_{G_1}(u) + S_{G_1}(v))(d_{G_1}(u) + d_{G_1}(v))^2 \\ &\quad -2M_1^4(G_2)\sum_{uv\in E(G_1)} (S_{G_1}(u) + S_{G_1}(v))(d_{G_1}(u) + d_{G_1}(v))^3. \end{aligned}$$

Proof. By definition of first neighbourhood Zagreb index, we have

$$\begin{aligned}
NM_1(G_1 \otimes_{T_1} G_2) &= \sum_{(u,v)\in V(G_1\otimes_{T_1} G_2)} S_{G_1\otimes_{T_1} G_2}(u,v)^2 \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_{T_1} G_2}(u,v)^2 + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_{T_1} G_2}(e,v)^2 \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big((S_{G_2}(v)(d^2_{G_1}(u) + S_{G_1}(u)))\Big)^2 \\
&\quad + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} \Big((d_{G_1}(u) + d_{G_1}(v))(S_{G_2}(v) \\
&\quad + d^2_{G_2}(v)(S_{G_1}(u) + S_{G_1}(v) - (d_{G_1}(u) + d_{G_1}(v)\Big)^2.
\end{aligned}$$

Expanding and by applying summation, we get

$$\begin{aligned}
NM_1(G_1 \otimes_{T_1} G_2) &= NM_1(G_2)M_1(G_1) + NM_1(G_1)NM_1(G_2) + NM_1(G_2)HM(G_1) \\
&\quad + M_1^4(G_2)\chi_4(G_1) + 2NM_1(G_2) \sum_{u\in V(G_1)} d^2_{G_1}(u)S_{G_1}(u) \\
&\quad + M_1^4(G_2) \sum_{uv\in E(G_1)} (S_{G_1}(u) + S_{G_1}(v))^2(d_{G_1}(u) + d_{G_1}(v))^2 \\
&\quad + 2 \sum_{v\in V(G_2)} S_2(v)d_{G_2}(v) \sum_{uv\in E(G_1)} (S_{G_1}(u) + S_{G_1}(v))(d_{G_1}(u) + d_{G_1}(v))^2 \\
&\quad - 2M_1^4(G_2) \sum_{uv\in E(G_1)} (S_{G_1}(u) + S_{G_1}(v))(d_{G_1}(u) + d_{G_1}(v))^3.
\end{aligned}$$

□

The following theorem gives the first neighbourhood Zagreb index of T-tensor products of two graphs G_1 and G_2.

Theorem 3.12. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned}
NM_1(G_1 \otimes_T G_2) &= 4HM(G_1)NM_1(G_2) + 4NM_1(G_1)NM_1(G_2) + M_1^4(G_2)\chi_4(G_1) \\
&\quad + NM_1(G_2) \sum_{u\in V(G_1)} (d^2_{G_1}(u) + S_{G_1}(u))^2 - 4\chi_3(G_1) \sum_{u\in V(G_2)} d^2_{G_2}(v)S_{G_2}(v) \\
&\quad + 4NM_1(G_2) \sum_{u\in V(G_1)} S_{G_1}(u)(d^2_{G_1}(u) + S_{G_1}(u)) \\
&\quad + M_1^4(G_2) \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2(S_{G_1}(u) + S_{G_1}(v))^2 \\
&\quad + 4 \sum_{v\in V(G_2)} S_{G_2}(v)d^2_{G_2}(v) \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2(S_{G_1}(u) + S_{G_1}(v)) \\
&\quad - 2M_1(G_2) \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2(S_{G_1}(u) + S_{G_1}(v)).
\end{aligned}$$

Proof. By definition of first neighbourhood Zagreb index, we have

$$NM_1(G_1 \otimes_T G_2) = \sum_{(u,v)\in V(G_1\otimes_T G_2)} S_{G_1\otimes_T G_2}(u,v)^2$$

$$\begin{aligned}
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_T G_2}(u,v)^2 + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} S_{G_1\otimes_T G_2}(e,v)^2 \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big(2S_{G_1}(u)S_{G_2}(v) + S_{G_2}(v)(d^2_{G_1}(u) + S_{G_1}(u)\Big)^2 \\
&\quad + \sum_{e=uv\in E(G_1)} \sum_{v\in V(G_2)} \Big(2S_{G_2}(v)(d_{G_1}(u) + d_{G_1}(v)) + d^2_{G_2}(v)(d_{G_1}(u) \\
&\quad + d_{G_1}(v))(S_{G_1}(u) + S_{G_1}(v) - (d_{G_1}(u) + d_{G_1}(v)))\Big)^2.
\end{aligned}$$

Expanding and by applying summation, we get

$$\begin{aligned}
NM_1(G_1 \otimes_T G_2) &= 4HM(G_1)NM_1(G_2) + 4NM_1(G_1)NM_1(G_2) + M_1^4(G_2)\chi_4(G_1) \\
&\quad + NM_1(G_2) \sum_{u\in V(G_1)} (d^2_{G_1}(u) + S_{G_1}(u))^2 - 4\chi_3(G_1) \sum_{u\in V(G_2)} d^2_{G_2}(v)S_{G_2}(v) \\
&\quad + 4NM_1(G_2) \sum_{u\in V(G_1)} S_{G_1}(u)(d^2_{G_1}(u) + S_{G_1}(u)) \\
&\quad + M_1^4(G_2) \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 (S_{G_1}(u) + S_{G_1}(v))^2 \\
&\quad + 4 \sum_{v\in V(G_2)} S_{G_2}(v) d^2_{G_2}(v) \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 (S_{G_1}(u) + S_{G_1}(v)) \\
&\quad - 2M_1(G_2) \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 (S_{G_1}(u) + S_{G_1}(v)).
\end{aligned}$$

□

4. **Conclusion**

In this paper, we have obtained first, second multiplicative Zagreb indices and first neighbourhood Zagreb index of $\mathcal{F}$-tensor products of graphs for each case $\mathcal{F} \in \{S, T_2, T_1, T\}$.

Acknowledgement. The second author is supported by Directorate of Minorities, Government of Karnataka, Bangalore, through M.Phil/Ph.D fellowship-2019-20:No.DOM/Ph.D/M.Phil/FELLOWSHIP/CR-01/2019-20 dated 15^{th} October 2019.

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