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FIXED POINT RESULTS IN COMPLEX VALUED EXTENDED b-METRIC SPACES AND RELATED APPLICATIONS

NAIMAT ULLAH[1] AND MOHAMMED SHEHU SHAGARI[2*]

Abstract. In this paper, some fixed point results of single-valued mappings satisfying a rational contraction in complex valued extended b-metric spaces are established. Nontrivial examples which support the hypotheses of the established idea herein are provided. Moreover, sufficient conditions for existence of a unique solution of nonlinear integral equation is considered to show a usability of the established concepts. Our result, being discussed in the setting of extended b-metric spaces, improves a few fixed point theorems in the framework of complex valued metric and b-metric spaces in the comparable literature.

1. Introduction

Fixed point theory is one of the most active research fields in modern nonlinear functional analysis. In general, fixed point problem is of the form $Tx = x$, where T is a self-mapping on a non-empty set X. This problem can be reformulated as $g(x) = 0$, where $g(x) = x - Tx$. As simple as this problem statement, finding its solution may be extremely difficult and sometimes not obtainable. The earliest affirmative response to this problem was presented by Banach [9] under some suitable conditions: when T is a contraction and X is equipped with a norm such that the corresponding topology yields completeness. So far, fixed point techniques have gained enormous applications in diverse areas such as biology, economics, chemistry, physics, engineering and so on. In 1969, Kannan[17] gave an analogue sort of contractive condition that demonstrated the existence of fixed point. The basic distinction between Banach fixed point theorem (BFT) and that of Kannan contraction is that continuity of contraction is not required in the latter. Similar well-known improvements of the BFT were established by Chatterjea [10] and Edelstein [13]. In the last five decades, the above results have been generalized in different directions. For a comprehensive survey on this subject, the interested reader may consult Rhoades [19].

It is well-known that fixed point theorems involving rational contractions cannot be extended or not worthwhile in cone metric spaces. To resolve this problem,

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* Corresponding author.

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Azam et al. [4] introduced the concept of complex valued metric spaces and established sufficient conditions for existence of common fixed points of a pair of mappings satisfying contractive type conditions involving rational expressions. It is interesting to note that complex valued metric space is a special class of cone metric spaces. But, the definition of a cone metric is based on the underlying Banach space which is not a division ring; thus, many results of analysis concerning divisions cannot be generalized to cone metric spaces. On this development, the study of fixed point theorems concerning rational inequalities in complex valued metric spaces have been growing vigorously (see, for example, [1, 2, 5, 6, 11, 12, 18, 21]). Along the line, the idea of b-metric space was presented by Bakhtin [8] in 1989. Rao [20] introduced the notion of fixed point results on complex valued b-metric spaces, which is broader than complex valued metric spaces. However, every complex valued b-metric space is a cone b-metric space over Banach algebra $\mathbb{C}$ in which the cone is normal with the coefficient of normality $K = 1$, and where the cone has non-empty interior (that is, solid cone). Following [20], various authors have demonstrated fixed point results for different mappings fulfilling rational inequalities with regards to complex valued b-metric spaces (see, for instance, [3, 7]).

In this paper, a fixed point result of single-valued mapping satisfying a rational contractive inequality in complex valued extended b-metric spaces is established. Nontrivial examples which support the hypotheses of the obtained concept herein are provided. Thereafter, sufficient conditions for existence of a unique solution of non-linear integral equation is considered to show a usability of the presented results. Our main ideas, being discussed in the setting of extended b-metric spaces, improves some known fixed point theorems in the setting of complex valued metric and b-metric spaces in the corresponding literature.

2. Preliminaries

In this section, we recall some specific concepts which are necessary for the presentation of our main results.

Definition 2.1. [4] Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. The partial order on $\mathbb{C}$ is defined as:
$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. This implies that $z_1 \preceq z_2$ if one of the below conditions is fulfilled:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$.

Definition 2.2. [4] Let $\mathbb{X}$ be a non-empty set. If the mapping $\phi : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ satisfies the following conditions :
(i) $0 \preceq \phi(x, y)$ and $\phi(x, y) = 0 \iff x = y$;
(ii) $\phi(x, y) = \phi(y, x)$;
(iii) $\phi(x, y) \preceq \phi(x, z) + \phi(z, y)$, $\forall\ x, z, y \in \mathbb{X}$,

then ϕ is known as a complex valued metric on $\mathbb{X}$, and the pair $(\mathbb{X}, \phi)$ is said to be a complex valued metric space.

Example 2.3. Let $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$, where

$$\mathbb{X}_1 = \{z \in \mathbb{C} : Re(z) \geq 0 \text{ and } Im(z) = 0\}$$

and

$$\mathbb{X}_2 = \{z \in \mathbb{C} : Re(z) = 0 \text{ and } Im(z) \geq 0\}.$$

Define $\phi : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{C}$ as follows:

$$\phi(z_1, z_2) = \begin{cases} \frac{2}{3}\left|x_1 - x_2\right| + \frac{i}{2}\left|x_1 - x_2\right|, & \text{if } z_1, z_2 \in \mathbb{X}_1; \\ \frac{1}{2}\left|y_1 - y_2\right| + \frac{i}{2}\left|y_1 - y_2\right|, & \text{if } z_1, z_2 \in \mathbb{X}_2; \\ \left(\frac{1}{2}y_1 + \frac{2}{3}x_2\right) + i\left(\frac{1}{3}y_1 + \frac{1}{2}x_2\right), & \text{if } z_1 \in \mathbb{X}_1, z_2 \in \mathbb{X}_2, \end{cases}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, $(\mathbb{X}, \phi)$ is a complex valued metric space.

Lemma 2.4. [4] *Let $(\mathbb{X}, \phi)$ be a complex valued metric space and $\{x_p\}_{p\geq 1}$ be a sequence in $\mathbb{X}$. Then $\{x_p\}_{p\geq 1}$ is said to converge to $x \in \mathbb{X}$ if and only if $|\phi(x_p, x)| \to 0$ as $p \to \infty$.*

Lemma 2.5. [4] *Let $(\mathbb{X}, \phi)$ be a complex valued metric space and $\{x_p\}_{p\geq 1}$ be a sequence in $\mathbb{X}$. Then $\{x_p\}_{p\geq 1}$ is a Cauchy sequence if and only if $|\phi(x_p, x_q)| \to 0$ as $p, q \to \infty$.*

Definition 2.6. [20] Let $\mathbb{X}$ be a non-empty set and $\tau \geq 1$ be a real number. A function $\phi : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ is called complex valued b-metric space, if for all $x, y, z \in \mathbb{X}$, the following conditions hold.
(i) $0 \preceq \phi(x, y)$ and $\phi(x, y) = 0$ if and only if $x = y$;
(ii) $\phi(x, y) = \phi(y, x)$;
(iii) $\phi(x, y) \preceq \tau[\phi(x, z) + \phi(z, y)]$.
Then ϕ is called a complex valued b-metric on $\mathbb{X}$ and the pair $(\mathbb{X}, \phi)$ is called a complex valued b-metric space.

Example 2.7. [20] Let $\mathbb{X} = [0, 1]$. Define a mapping $\phi : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ by

$$\phi(x, y) = |x - y|^2 + i|x - y|^2$$

for all $x, y \in \mathbb{X}$. Then $(\mathbb{X}, \phi)$ is a complex valued b-metric space with $\tau = 2$.

Definition 2.8. Let $\mathbb{X}$ be a non-empty set and $\theta : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ be a function. Then the function $\phi : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ is known as a complex valued extended b-metric if the following are satisfied for all $x, y, z \in \mathbb{X}$:
(i) $0 \preceq \phi(x, y)$ and $\phi(x, y) = 0$ if and only if $x = y$;
(ii) $\phi(x, y) = \phi(y, x)$;
(iii) $\phi(x, z) \preceq \theta(x, z)[\phi(x, y) + \phi(y, z)]$.
Then the pair $(\mathbb{X}, \phi)$ is known as a complex valued extended b-matric space.

Example 2.9. [21] Let $\mathbb{X} = \mathbb{C}$ be the set of all complex numbers and the function $\theta : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ be defined as:

$$\theta(x, y) = \frac{1 + x + y}{x + y}$$

.
Further, Let

(i) $\phi(x,y) = \frac{i}{xy}$, for all $x, y \in (0,1]$;
(ii) $\phi(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in [0,1]$;
(iii) $\phi(x,0) = \phi(0,x) = \frac{i}{x}$ for all $x \in (0,1]$.

Then the pair $(\mathbb{X}, \phi)$ is a complex valued extended b-metric space.

Example 2.10. Let $\mathbb{X} = [0,\infty)$. $\theta : \mathbb{X} \times \mathbb{X} \to [1,\infty)$ be a function defined by $\theta(x,y) = 1 + x + y$ and $\phi : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{C}$ be given as

$$\phi(x,y) = \begin{cases} 0, & \text{if} \quad x = y \\ i, & \text{if} \quad x \neq y. \end{cases}$$

Then $(\mathbb{X}, \phi)$ is a complex valued extended b-metric space.

3. Main results

Our main result is as follows.

Theorem 3.1. *Let $(\mathbb{X}, \phi)$ be a complete complex valued extended b-metric space, $\theta : X \times X \to [1,\infty)$ a function and let A and B be self-mappings on $\mathbb{X}$ satisfying:*

$$\phi(Ax, By) \preceq \frac{2\alpha}{\theta(x,y)} \psi(x,y), \tag{3.1}$$

for all $x, y \in \mathbb{X}$, where $\alpha \in (0,1)$,

$$\psi(x,y) = \max\left\{ \phi(x,y), \frac{\phi(x,Ax)\phi(y,By)}{1 + \phi(x,y)}, \frac{\phi(x,Ay)\phi(y,Bx)}{1 + \phi(x,y)} \right\},$$

and assumed that $\theta(x_n, x_{n+1}) < \theta(x_n, x_{n+2})$ as $n \longrightarrow \infty$, for every sequence $\{x_n\}_{n\geq 1}$ in $\mathbb{X}$.
Then there exists a unique common fixed point of the pair (A, B).

Proof. Let $x_0 \in \mathbb{X}$ be an arbitrary element and define a sequence $\{x_n\}_{n\geq 1}$ as follows:

$$x_{2n+1} = Ax_{2n} \text{ and } x_{2n+2} = Bx_{2n+1}, \; n = 0, 1, 2, ... \tag{3.2}$$

Then, using equations (3.1) and (3.2), we have

$$\begin{aligned}
&\phi(x_{2n+1}, x_{2n+2}) = \phi(Ax_{2n}, Bx_{2n+1})\\
&\preceq \frac{2\alpha}{\theta(x_{2n}, x_{2n+1})}\\
&\times \max\left\{\phi(x_{2n}, x_{2n+1}), \frac{\phi(x_{2n}, Ax_{2n})\phi(x_{2n+1}, Bx_{2n+1})}{\phi(x_{2n}, x_{2n+1})}, \frac{\phi(x_{2n}, Bx_{2n+1}))\phi(x_{2n+1}, Ax_{2n})}{\phi(x_{2n}, x_{2n+1})}\right\}\\
&\preceq \frac{2\alpha}{\theta(x_{2n}, x_{2n+1})}\\
&\times \max\left\{\phi(x_{2n}, x_{2n+1}), \frac{\phi(x_{2n}, x_{2n+1})\phi(x_{2n+1}, x_{2n+2})}{\phi(x_{2n}, x_{2n+1})}, \frac{\phi(x_{2n}, x_{2n+2}))\phi(x_{2n+1}, x_{2n+1})}{\phi(x_{2n}, x_{2n+1})}\right\}\\
&\preceq \frac{2\alpha}{\theta(x_{2n}, x_{2n+1})}\\
&\times \max\left\{\phi(x_{2n}, x_{2n+1}), \phi(x_{2n+1}, x_{2n+2})\right\}.
\end{aligned}$$

If $\max\{\phi(x_{2n}, x_{2n+1}), \phi(x_{2n+1}, x_{2n+2})\} = \phi(x_{2n+1}, x_{2n+2})$, then

$$\phi(x_{2n+1}, x_{2n+2}) \preceq \frac{2\alpha}{\theta(x_{2n}, x_{2n+1})}\, \phi(x_{2n+1}, x_{2n+2}),$$

from which we have

$$\phi(x_{2n+1}, x_{2n+2})\left(1 - \frac{2\alpha}{\theta(x_{2n}, x_{2n+1})}\right) \preceq 0,$$

that is, $\phi(x_{2n+1}, x_{2n+2}) = 0$, which is not possible. So,

$$\phi(x_{2n+1}, x_{2n+2}) \preceq \frac{2\alpha}{\theta(x_{2n}, x_{2n+1})}\, \phi(x_{2n}, x_{2n+1}). \tag{3.3}$$

From (3.3), for all $n = 0, 1, 2....$, we write

$$\begin{aligned}
\phi(x_{n+1}, x_{n+2}) &\preceq \frac{2\alpha}{\theta(x_n, x_{n+1})}\, \phi(x_n, x_{n+1})\\
&\preceq \frac{(2\alpha)^2}{\theta(x_n, x_{n+1})\theta(x_{n-1}, x_n)}\, \phi(x_{n-1}, x_n)\\
&\preceq \frac{(2\alpha)^3}{\theta(x_n, x_{n+1})\theta(x_{n-1}, x_n)\theta(x_{n-2}, x_{n-1})}\, \phi(x_{n-2}, x_{n-1})\\
&.\\
&.\\
&.\\
&\preceq \frac{(2\alpha)^{n+1}}{\theta(x_{n+1}, x_{n+2})\theta(x_n, x_{n+1})\theta(x_{n-1}, x_n)...\theta(x_2, x_3)\theta(x_1, x_2)}\phi(x_0, x_1)
\end{aligned}$$

By using triangular inequality for extended b-metric space, for $m > n$, we have

$$\phi(x_n, x_m)$$
$$\preceq \theta(x_n, x_m)\phi(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\phi(x_{n+1}, x_{n+2}) + \ldots +$$
$$\theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)\phi(x_{m-1}, x_m)$$
$$\preceq \left\{\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)(2\alpha)^n}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_{n-1}, x_n)\theta(x_n, x_{n+1})\theta(x_{n+1}, x_{n+2})}\right\}\phi(x_0, x_1)$$
$$+\left\{\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\theta(x_{n+1}, x_m)(2\alpha)^{n+1}}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_{n-1}, x_n)\theta(x_n, x_{n+1})\theta(x_{n+1}, x_{n+2})}\right\}\phi(x_0, x_1)$$
$$+\left\{\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)(2\alpha)^{n+2}}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_n, x_{n+1})\theta(x_{n+1}, x_{n+2})\theta(x_{n+2}, x_{n+3})}\right\}\phi(x_0, x_1)$$

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$$\left\{\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)(2\alpha)^{m-1}}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_n, x_{n+1})\ldots\theta(x_{m-2}, x_{m-1})\theta(x_{m-1}, x_m)}\right\}\phi(x_0, x_1)$$

The above expression gives

$$\phi(x_n, x_m)$$
$$\preceq$$
$$\phi(x_0, x_1)\Bigg[\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)(2\alpha)^n}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_{n-1}, x_n)\theta(x_n, x_{n+1})\theta(x_{n+1}, x_{n+2})}$$
$$+\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\theta(x_{n+1}, x_m)(2\alpha)^{n+1}}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_{n-1}, x_n)\theta(x_n, x_{n+1})\theta(x_{n+1}, x_{n+2})}$$

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$$+\frac{\theta(x_1, x_m)\theta(x_2, x_m)\ldots\theta(x_n, x_m)\theta(x_{n+1}, x_m)\ldots\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)(2\alpha)^{m-1}}{\theta(x_1, x_2)\theta(x_2, x_3)\ldots\theta(x_n, x_{n+1})\theta(x_{n+1}, x_{n+2})\ldots\theta(x_{m-2}, x_{m-1})\theta(x_{m-1}, x_m)}\Bigg]$$

By hypothesis, $\theta(x_n, x_{n+1}) < \theta(x_n, x_{n+2})$ as $n \longrightarrow \infty$, so taking $m = n+2$ for all $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{m-1}\theta(x_i, x_m)}{\prod_{i=1}^{m-1}\theta(x_i, x_{i+1})}$ converges by ratio test. For $m \in \mathbb{N}$, let

$$S = \sum_{m=2}^{\infty} \frac{(2\alpha)^{m-1}\prod_{i=1}^{m-1}\theta(x_i, x_m)}{\prod_{i=1}^{m-1}\theta(w_i, w_{i+1})},\ S_n = \sum_{j=1}^{n} \frac{(2\alpha)^j\prod_{i=1}^{j}\theta(x_i, x_m)}{\prod_{i=1}^{j}\theta(x_i, x_{i+1})}.$$

Then, fore for $m > n$, the above inequality implies that

$$\phi(x_n, x_m) \preceq \phi(x_0, x_1)\big[S_{m-1} - S_n\big].$$

By letting $n \to \infty$ in the above inequality, $\phi(x_n, x_m) \preceq 0$, therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{X}$. Since $\mathbb{X}$ is complete, So $x_n \to u \in \mathbb{X}$ as $n \to \infty$. If A and B are not continuous, then $u = Au$, otherwise, $\phi(u, Au) = z > 0$ and one gets

$$
\begin{aligned}
& z \\
& \preceq \theta(u,u)\{\phi(u, x_{2p+1}) + \phi(x_{2p+2}, Au)\} \\
& \preceq \theta(u,u)\{\phi(u, x_{2p+1}) + \phi(Au, Bx_{2p+1})\} \\
& \preceq \theta(u,u)\phi(u, x_{2p+1}) \\
& + \frac{2\alpha\theta(u,u)}{\theta(u, x_{2p+1})} \max\left\{\phi(u, x_{2p+1}), \frac{\phi(u, Au)\phi(x_{2p+1}, Bx_{2p+1})}{1+\phi(u, x_{2p+1})}, \frac{\phi(u, Bx_{2p+1})\phi(x_{2p+1}, Au)}{1+\phi(u, x_{2p+1})}\right\} \\
\preceq & \theta(u,u)\phi(u, x_{2p+1}) \\
& + \frac{2\alpha\theta(u,u)}{\theta(u, x_{2p+1})} \max\left\{\phi(u, x_{2p+1}), \frac{\phi(u, Au)\phi(x_{2p+1}, x_{2p+2})}{1+\phi(u, x_{2p+1})}, \frac{\phi(u, x_{2p+2})\phi(x_{2p+1}, Au)}{1+\phi(u, x_{2p+1})}\right\} \\
\preceq & \theta(u,u)\phi(u, x_{2p+1}) \\
& + \frac{2\alpha\theta(u,u)}{\theta(u, x_{2p+1})} \max\left\{\phi(u, Ax_{2p}), \frac{\phi(u, Au)\phi(x_{2p+1}, x_{2p+2})}{1+\phi(u, x_{2p+1})}, \frac{\phi(u, x_{2p+2})\phi(x_{2p+1}, Au)}{1+\phi(u, x_{2p+1})}\right\}
\end{aligned}
$$

By taking limit as $p \to \infty$ in the above expression, we have

$$
\begin{aligned}
z \preceq & \theta(u,u)\phi(u,u) + \frac{2\alpha\theta(u,u)}{\theta(u,u)} \max\left\{\phi(u, Au), \frac{\phi(u, Au)\phi(u,u)}{1+\phi(u,u)}, \frac{\phi(u,u)\phi(u, Au)}{1+\phi(u,u)}\right\} \\
z \preceq & 0 + 2\alpha \max\{\phi(u, Au), 0, 0\} \\
z \preceq & 0 + 2\alpha \max\{z, 0, 0\}.
\end{aligned}
$$

The above expression gives $|z(1-2\alpha)| \leq 0$; which implies that $|z| \leq 0$. So, the only possibility is $|z| = 0$, which is a contradiction. Hence $u = Au$. Similarly, we can show that $u = Bu$. If A and B are continuous, then

$$
u = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Ax_{2n+1} = A \lim_{n\to\infty} x_{2n+1} = Au.
$$

On similar steps, $u = Bu$. Hence, the pair (A, B) has a common fixed point in $\mathbb{X}$. Uniqueness: Suppose that u^* is another fixed point of the mappings A and B, then

$$
\begin{aligned}
\phi(u, u^*) &= \phi(Au, Bu^*) \\
&\preceq \frac{2\alpha}{\theta(u, u^*)} \max\left\{\phi(u, u^*), \frac{\phi(u, Au)\phi(u^*, Bu^*)}{1+\phi(u, u^*)}, \frac{\phi(u, Bu^*)\phi(u^*, Au)}{1+\phi(u, u^*)}\right\} \\
&\preceq \frac{2\alpha}{\theta(u, u^*)}\phi(u, u^*).
\end{aligned}
$$

From the above inequality, $\phi(u,u^*)\left[1-\frac{2\alpha}{\theta(u,u^*)}\right] \precsim 0$; which implies that $|\phi(u,u^*)|\left|1-\frac{2\alpha}{\theta(u,u^*)}\right| \leq 0$. Since $\left|1-\frac{2\alpha}{\theta(u,u^*)}\right| \neq 0$, then the only possibility is $|\phi(u,u^*)| = 0$. Consequently, $u = u^*$. □

Example 3.2. Let $\mathbb{X} = [0,1)$ and define $\phi : \mathbb{X}\times\mathbb{X} \to \mathbb{C}$ by:

$$\phi(x,y) = i|x-y| \text{ for all } x,y \in \mathbb{X}.$$

Let $\theta : \mathbb{X}\times\mathbb{X} \to [1,\infty)$ be a function defined as:

$$\theta(x,y) = 1+x+y,$$

then $(\mathbb{X},\phi)$ is a complete complex valued extended b-metric space. Define the self-mappings A and B on $\mathbb{X}$ as

$$Ax = \frac{1}{10}(x^2+x) \text{ and } By = \frac{1}{5}y^2+y^3.$$

Then the contraction condition of the main theorem is satisfied for all x,$y \in \mathbb{X}$ with $\theta(x,y) = 1+x+y = \frac{11}{6}$ and $2\alpha = 1.8$, for $\alpha = 0.9$. We see that $0 \in \mathbb{X}$ is the unique common fixed of A and B.

Taking $A = B$ in Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $(\mathbb{X},\phi)$, be a complete complex valued extended b-metric space, $\theta : X\times X \to [1,\infty)$ be a function and let A be self-mapping on $\mathbb{X}$ satisfying*

$$\phi(Ax,Ay) \precsim \frac{2\alpha}{\theta(x,y)}\psi(x,y), \tag{3.4}$$

for all $x,y \in \mathbb{X}$, where $\alpha \in (0,1)$,

$$\psi(x,y) = \max\left\{\phi(x,y), \frac{\phi(x,Ax)\phi(y,Ay)}{1+\phi(x,y)}, \frac{\phi(x,Ay)\phi(y,Ax)}{1+\phi(x,y)}\right\},$$

and assumed that $\theta(x_n,x_{n+1}) < \theta(x_n,x_{n+2})$, for every sequence $\{x_n\}_{n\geq 1}$ in $\mathbb{X}$. Then there exists a unique fixed point of A.

For the verification of Corollary 3.3, the following example is given:

Example 3.4. Let $\mathbb{X} = [0,1)$ and define $\phi : \mathbb{X}\times\mathbb{X} \to \mathbb{C}$ by:

$$\phi(x,y) = \begin{cases} 0, & \text{if} \quad x = y \\ i, & \text{if} \quad x \neq y. \end{cases}$$

$\theta : \mathbb{X}\times\mathbb{X} \to [1,\infty)$ be a function defined by:

$$\theta(x,y) = 1+x+y,$$

then $(\mathbb{X},\phi)$ is a complete complex valued extended b-metric space.
Define a self mapping A by $Ax = \frac{1}{3}(2-x)$. Then, it is easy to verify that all the conditions of Corollary 3.3 are satisfied, and hence $\frac{1}{2}$ is the fixed point of A.

Corollary 3.5. *Let $(\mathbb{X}, \phi)$ be a complete complex valued extended b-metric space, $\theta : X \times X \to [1, \infty)$ be a function and let A be a self-mapping on $\mathbb{X}$ such that for each $n \in \mathbb{N}$,*

$$\phi(A^n x, A^n y) \preceq \frac{2\alpha}{\theta(x,y)} \psi(x,y), \tag{3.5}$$

for all $x, y \in \mathbb{X}$, where $\alpha \in (0,1)$,

$$\psi(x,y) = \max\left\{ \phi(x,y), \frac{\phi(x, A^n x)\phi(y, A^n y)}{1+\phi(x,y)}, \frac{\phi(x, A^n y)\phi(y, A^n x)}{1+\phi(x,y)} \right\},$$

and assumed that $\theta(x_n, x_{n+1}) < \theta(x_n, x_{n+2})$ for every sequence $\{x_n\}_{n\geq 1}$ in $\mathbb{X}$. Then there exists a unique fixed point of A.

Proof. By Corollary 3.3, we have $z \in \mathbb{X}$ such that

$A^n z = z$. Therefore,

$$\begin{aligned}
\phi(Az, z) &= \phi(AA^n z, A^n z) = \phi(A^n Az, A^n z) \\
&\preceq \frac{2\alpha}{\theta(Az,z)} \max\left\{ \phi(Az,z), \frac{\phi(Az, A^n Az)\phi(z, A^n z)}{1+\phi(Az,z)} \frac{\phi(Az, A^n z)\phi(z, A^n Az)}{1+\phi(Az,z)} \right\} \\
&= \frac{2\alpha}{\theta(Az,z)} \max\left\{ \phi(Az,z), \frac{\phi(Az, AA^n z)\phi(z, A^n z)}{1+\phi(Az,z)} \frac{\phi(Az, A^n z)\phi(z, AA^n z)}{1+\phi(Az,z)} \right\} \\
&= \frac{2\alpha}{\theta(Az,z)} \max\left\{ \phi(Az,z), \frac{\phi(Az, Az)\phi(z, z)}{1+\phi(Az,z)} \frac{\phi(Az, z)\phi(z, Az)}{1+\phi(Az,z)} \right\} \\
&= \frac{2\alpha}{\theta(Az,z)} \max\left\{ \phi(Az,z), 0, \frac{(\phi(z, Az))^2}{1+\phi(Az,z)} \right\} \\
&\preceq \frac{2\alpha}{\theta(Az,z)} \phi(Az,z) \\
&\preceq \frac{2\alpha}{\theta(Az,z)} \phi(Az,z).
\end{aligned}$$

The above expression implies that $\left|1 - \frac{2\alpha}{\theta(Az,z)}\right| |\phi(Az,z)| \leq 0$. Following previous arguments, we conclude that $Az = z$. □

The following example shows an application of Corollary 3.5.

Example 3.6. Find a solution of the equation:

$$\frac{dx}{dt} = xe^{t-1}, \ 1 \leq t \leq 3, \ x(1) = 4. \tag{3.6}$$

Proof. The integral reformulation of (3.6) is given by:

$$x(t) = 4 + \int_1^t x(u)e^{u-1} du. \tag{3.7}$$

Consider a metric space $\mathbb{X} = C([1,3],\mathbb{R})$, where $C([1,3],\mathbb{R})$ is the space of all continuous function defined on $[1,3]$. Define $w : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ by

$$w(x,y) = \max_{1\leq t\leq 3} |x(t) - y(t)| e^{i(t-1)}.$$

Clearly, $(\mathbb{X}, w)$ is a complete complex valued extended b-metric space. Define $A : \mathbb{X} \to \mathbb{X}$ as follows:

$$Ax(t) = 4 + \int_1^t x(u)e^{u-1}du.$$

Put $\theta(x,y) = 1$, then

$$\begin{aligned} w(Ax(t), Ay(t)) &= \max_{1\leq t\leq 3} |Ax(t) - Ay(t)| e^{i(t-1)} \\ &= \max_{1\leq t\leq 3} \left| \int_1^t [x(u) - y(u)] e^{u-1} \right| . e^{i(t-1)} du \\ &\preceq \max_{1\leq t\leq 3} \int_1^t |x(u) - y(u)| e^{i(t-1)} |e^{u-1}| du \\ &= \int_1^t \max_{1\leq t\leq 3} |x(u) - y(u)| e^{i(t-1)} . \max_{1\leq t\leq 3} |e^{u-1}| du \\ &= (t-1)e^2 w(x,y). \end{aligned}$$

Similarly,

$$\begin{aligned} w(A^n x, A^n y) &\preceq e^{2n} \frac{(t-1)^n}{n!} w(x,y) \\ &\preceq e^{2n} \frac{(2)^n}{n!} w(x,y), \end{aligned}$$

where

$$\frac{2\alpha}{\theta(x,y)} = e^{2n} \frac{(2)^n}{n!} = \begin{cases} 8.3643, & \text{if } n = 35 \\ 3.4335, & \text{if } n = 36 \\ 1.3714, & \text{if } n = 37 \\ 0.5333, & \text{if } n = 38. \end{cases}$$

Hence, all the conditions of Corollary 3.5 are verified. Thus, A has a unique fixed point in $\mathbb{X}$, which is the unique fixed of the differential equation 3.6. □

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[1]DEPARTMENT OF MATHEMATICS, VISITING FACULTY, UNIVERSITY OF MIANWALI, PAKISTAN

Email address: naimat0347@gmail.com

[2]DEPARTMENT OF MATHEMATICS, FACULTY OF PHYSICAL SCIENCES, AHMADU BELLO UNIVERSITY, NIGERIA

Email address: shagaris@ymail.com